Thinking about Mixed-effects Models

Andrew Robinson

Department of Mathematics & Statistics
University of Melbourne

February 17, 2006
1 The Deep End

- Model Statement
- MLE
- ReML
Basic Model Statement

\[ Y = X\beta + Zb + \epsilon \]

\[ b \sim \mathcal{N}(0, D) \]
\[ \epsilon \sim \mathcal{N}(0, R) \]
Basic Model Statement

\[ Y = X\beta + Zb + \epsilon \]

\[ b \sim \mathcal{N}(0, D) \]
\[ \epsilon \sim \mathcal{N}(0, R) \]

**Design Matrices**

- **X** allocates the fixed effects.
- **Z** allocates the random effects.
Basic Model Statement

\[ Y = X\beta + Zb + \epsilon \]

\[ b \sim \mathcal{N}(0, D) \]
\[ \epsilon \sim \mathcal{N}(0, R) \]

**Design Matrices**
- \( X \) allocates the fixed effects.
- \( Z \) allocates the random effects.

**Covariance Matrices**
- \( D \) describes the random effects covariance.
- \( R \) allocates the residuals covariance.
$Y = X\beta + Zb + \epsilon$

$\text{Var} (Y \mid X, Z, \beta, b) = R$
The Deep End

Model Statement
MLE
ReML

\[ Y = X_\beta + Zb + \epsilon \]

\[
\text{Var} (Y \mid X, Z, \beta, b) = R
\]

\[
\text{Var} (Y \mid X, \beta) = ZDZ' + R = V
\]
Log Likelihood

\[ \mathcal{L}(\beta, V \mid Y, X) = -\frac{1}{2} \ln (|V|) - \frac{n}{2} \ln (2\pi) - \frac{1}{2} (Y - X\beta)' V^{-1} (Y - X\beta) \]
Profile $\beta$ out

\[ \hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y \]
\[ \mathcal{L}(\beta, V \mid Y, X) = f(V, Y, X, Z, D, R) \]
Estimate $\hat{V}$ by maximization and then $\hat{\beta}$ by substitution.
Maximum likelihood estimators of covariance parameters are usually negatively biased.
Briefly, ReML involves applying ML, but replacing

- $Y$ with $KY$;
- $X$ with $0$;
- $Z$ with $K'Z$; and
- $V$ with $K'VK$

where $K$ is such that $K'X = 0$. 