Thinking about Mixed-effects Models

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February 17, 2006

Outline

Basic Model Statement

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{b} + \epsilon$$

$$\begin{aligned} \mathbf{b} &\sim \mathcal{N}(\mathbf{0}, \mathbf{D}) \\ \epsilon &\sim \mathcal{N}(\mathbf{0}, \mathbf{R}) \end{aligned}$$

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Covariance Matrices

- **D** describes the random effects covariance.
- **R** allocates the residuals covariance.

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$$\text{Var}\left(\mathbf{Y} \mid \mathbf{X}, \beta\right) = \mathbf{Z}\mathbf{D}\mathbf{Z}' + \mathbf{R} = V$$

Log Likelihood

$$\mathcal{L}\left(\beta, \mathbf{V} \mid \mathbf{Y}, \mathbf{X}\right) = -\frac{1}{2}\ln\left(|\mathbf{V}|\right) - \frac{n}{2}\ln\left(2\pi\right) - \frac{1}{2}\left(\mathbf{Y} - \mathbf{X}\beta\right)' \mathbf{V}^{-1}\left(\mathbf{Y} - \mathbf{X}\beta\right)$$

Profile β out

$$\hat{\beta} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$$

β IS GONE!

$$\mathcal{L}\left(\beta, \mathbf{V} \mid \mathbf{Y}, \mathbf{X}\right) = f\left(\mathbf{V}, \mathbf{Y}, \mathbf{X}, \mathbf{Z}, \mathbf{D}, \mathbf{R}\right)$$

β is gone!

Estimate $\hat{V}$ by maximization and then $\hat{\beta}$ by substitution.

ReML

Maximum likelihood estimators of covariance parameters are usually negatively biased.

ReML

Briefly, ReML involves applying ML, but replacing

- $\mathbf{Y}$ with $\mathbf{KY}$;
- $\mathbf{X}$ with $\mathbf{0}$;
- $\mathbf{Z}$ with $\mathbf{K'Z}$; and
- $\mathbf{V}$ with $\mathbf{K'VK}$

where $\mathbf{K}$ is such that $\mathbf{K'X} = 0$.