Generalized Least Squares (GLS) and Estimated Generalized Least Squares (EGLS)

Linear Model in matrix notation for the population

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \mathrm{Var}(\boldsymbol{\varepsilon}) = \sum$$

<u>In GLS</u>, the error covariance matrix is known

<u>In EGLS</u> it is estimated from the sample data.

For GLS (and for EGLS), <u>as with OLS</u>:

- Y is a continuous variable; (n X 1) matrix
  - There is one y-variable (can be extended to more than one by stacking the y-variables—covered later)
- X: fixed-effect matrix (explanatory variables); (n X p) for "m=p-1" explanatory variables
  - "fixed" , and *x* is measured without error (more on this later)
  - The first column of the X matrix is all ones, for the intercept of the equation.
  - continuous variables, and/or class variables represented by dummy (indicator) variables
  - If there is more than one 1 *y* variable, then the matrix of *x*'s for each *y* variable would be stacked
- The coefficients are the "multipliers" for the x variables
  - The first coefficient is the intercept
  - The remaining coefficients are multiplied by one of the explanatory variables (are slopes).
  - If you fitted several equations simultaneously, the coefficients would be stacked to match with the X variables matrix.
- There is a **single error term (n X 1) matrix**
  - is a random component
  - Error terms follow a normal distribution
  - Not necessarily equal variances nor independent error terms.

The most general error covariance matrix is:

$$\sum = \begin{bmatrix} \sigma^2_{\varepsilon_1} & \sigma_{\varepsilon_1\varepsilon_2} & \sigma_{\varepsilon_1\varepsilon_3} & \cdots & \sigma_{\varepsilon_1\varepsilon_n} \\ \sigma_{\varepsilon_2\varepsilon_1} & \sigma^2_{\varepsilon_2} & \sigma_{\varepsilon_2\varepsilon_1} & \cdots & \sigma_{\varepsilon_2\varepsilon_n} \\ \sigma_{\varepsilon_3\varepsilon_1} & \sigma_{\varepsilon_3\varepsilon_2} & \sigma^2_{\varepsilon_3} & \cdots & \sigma_{\varepsilon_3\varepsilon_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{\varepsilon_n\varepsilon_1} & \sigma_{\varepsilon_n\varepsilon_2} & \sigma_{\varepsilon_n\varepsilon_3} & \cdots & \sigma^2_{\varepsilon_n} \end{bmatrix}$$

- Variances of each error term are on the diagonal
- Covariances between error terms are on the off-diagonals
- This type of error covariance matrix is called an *unstructured* error covariance matrix. [no restrictions]

Under ordinary least squares (OLS), error terms are independent and identically distributed as the normal distribution (iid, normal). The error covariance matrix then becomes very simple:

$$\Sigma = \begin{bmatrix} \sigma_\varepsilon^{\,2} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_\varepsilon^{\,2} & 0 & \cdots & 0 \\ 0 & 0 & \sigma_\varepsilon^{\,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_\varepsilon^{\,2} \end{bmatrix} = I_n \sigma_\varepsilon^{\,2}$$

For this simpler matrix, we only need to know the variance of the errors, as a single value, to set up the entire error covariance matrix.

Properties of the Estimators for GLS and EGLS:

For OLS, since we only need to obtain an *unbiased estimator* for the variance of the errors (the MSE), we can obtain *unbiased* estimates of the coefficients and the standard errors of the coefficients. These will also be *consistent*. Assuming the errors also follow a normal distribution, these estimated coefficients are also *maximum likelihood*, and *Best Linear Unbiased Estimators (BLUE).*

Therefore, if we can make some transformations to the variables to obtain a linear relationship with the desired characteristics, we gain some desirable statistical properties.

For GLS and EGLS, the error covariance matrix differs from that of OLS.

GLS has the same small and large sample properties as OLS.

Since we almost never know the error covariance matrix, we almost never can use GLS. In practice, we loose some small sample properties when we need to use EGLS instead of OLS.

However, EGLS:

- Does have the small sample properties of unbiased estimators for the coefficients
- Does NOT have the small sample properties or unbiased estimators for the standard errors of these estimated coefficients NOR the small sample properties of minimum variance for estimators of the estimated coefficients
- If we obtain a consistent estimator of the error covariance matrix, then, EGLS has the large sample properties of:
    - Consistent estimate of the standard errors of the estimated coefficients
    - Asymptotically normally distributed
    - Asymptotically efficient estimates of the estimated coefficients
- Not possible to get a consistent estimator for "unstructured" error covariance matrix – too many elements to estimate from only "n" observations [may be nearly or asymptotically consistent estimators, such as that used in White's test, however]
- **In most circumstances, we must restrict the error covariance matrix from the "unstructured form" to get a consistent estimator used in fitting the equation based on EGLS.**

Common structures are:

1. Under the assumptions of unequal variance, no correlation, this matrix becomes:

$$\Sigma = \begin{bmatrix} \sigma_{\varepsilon_1}{}^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{\varepsilon_2}{}^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{\varepsilon_3}{}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varepsilon_n}{}^2 \end{bmatrix}$$

2. Under the assumptions of equal variances, and first order correlation (AR(1)), this matrix becomes:

$$\Sigma = \frac{\sigma_v{}^2}{(1-\rho)} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ 0 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix}$$

Since the equation becomes:

$$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \ldots + \beta_m x_{mt} + \varepsilon_t$$

and:

$$\varepsilon_t = \rho \varepsilon_{t-1} + v_t$$

$$v_t \sim N(0, \sigma_v{}^2), \text{ and are i.i.d.}$$

3. Another common pattern for the error covariance matrix is when we have groups of observations (grouped by subjects), which we will look at later on under maximum likelihood methods.

Estimates of the Coefficients

$$\mathbf{y} = \mathbf{x}\hat{\boldsymbol{\beta}} + \mathbf{e}$$

Using a sample of *n* observations from the population, where $\hat{\boldsymbol{\beta}}$ is the vector of coefficients (estimated parameters), and $\mathbf{e}$ is the vector of estimated errors (residuals) using this vector of estimated coefficients.

The generalized least squares (GLS) estimator of $\boldsymbol{\beta}$ is:

$$\hat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{y}$$

Usually, the error covariance matrix is also an estimate from the sample (want this to be a consistent estimate, in order to obtain some of the desirable large sample properties), and this is then called an estimated generalized least squares (EGLS) estimator of $\boldsymbol{\beta}$:

$$\hat{\boldsymbol{\beta}}_{EGLS} = (\mathbf{X}'\hat{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\Sigma}^{-1}\mathbf{y}$$

Under the assumptions of equal variance, no correlation, the error covariance matrix becomes very simple:

$$\Sigma = I_n \sigma_\varepsilon^{\;2}$$

where we only need to estimate $\sigma_\varepsilon^{\;2}$.

The EGLS estimator of $\boldsymbol{\beta}$ becomes simply the familiar least squares (called the ordinary least squares (or OLS)) estimate:

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{EGLS} &= (\mathbf{X}'\hat{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\Sigma}^{-1}\mathbf{y} \\
\hat{\boldsymbol{\beta}}_{EGLS} &= (\mathbf{X}'\left(I_n\hat{\sigma}^2\right)^{-1}\mathbf{X})^{-1}\mathbf{X}'\left(I_n\hat{\sigma}^2\right)^{-1}\mathbf{y} \\
\hat{\boldsymbol{\beta}}_{EGLS} &= (\mathbf{X}'I_n\left(\hat{\sigma}^2\right)^{-1}\mathbf{X})^{-1}\mathbf{X}'I_n\left(\hat{\sigma}^2\right)^{-1}\mathbf{y} \\
\hat{\boldsymbol{\beta}}_{EGLS} &= \left(\left(\hat{\sigma}^2\right)^{-1}\right)^{-1}\left(\hat{\sigma}^2\right)^{-1}(\mathbf{X}'I_n\mathbf{X})^{-1}\mathbf{X}'I_n\mathbf{y} \\
\hat{\boldsymbol{\beta}}_{EGLS} &= \frac{\hat{\sigma}^2}{\hat{\sigma}^2}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \hat{\boldsymbol{\beta}}_{OLS}
\end{aligned}$$

For simple linear regression with one explanatory variable, this gives the familiar sum of products XY divided by sum of squares of X for the slope.

Estimates of the Variances of the Estimated Coefficients

Using a sample of n observations from the population, we can estimate how much $\hat{\boldsymbol{\beta}}$ will vary from sample to sample (even though we only have one sample set of data from the population). The generalized least squares (GLS) estimator of the variance of $\hat{\boldsymbol{\beta}}_{GLS}$ is:

$$Var(\hat{\boldsymbol{\beta}}_{GLS}) = (\mathbf{X'}\Sigma^{-1}\,\mathbf{X})^{-1}$$

For the estimated generalized least squares (EGLS) estimator of the variance of $\hat{\boldsymbol{\beta}}_{EGLS}$:

$$V\hat{a}r(\hat{\boldsymbol{\beta}}_{EGLS}) = (\mathbf{X'}\hat{\Sigma}^{-1}\,\mathbf{X})^{-1}$$

Under the assumptions of equal variance, no correlation, the error covariance matrix becomes very simple, reduced to ordinary least squares (OLS) and we can calculate the estimator for variance of $\hat{\boldsymbol{\beta}}$ as:

$$V\hat{a}r(\hat{\boldsymbol{\beta}}_{OLS}) = \hat{\sigma}_{\varepsilon}^{\;2}(\mathbf{X'}\,\mathbf{X})^{-1}$$

In all cases, it is simply the first part of the corresponding equation used to estimate the coefficients.

Estimates of the Variances of the Mean Predicted y's

Once the estimated variances/covariances of the estimated coefficients are obtained, the variances for a mean predicted y can be obtained in a similar approach to that for OLS:

$$\hat{y}\,|\,\mathbf{x}_h = \hat{\mathbf{X}}_h\hat{\boldsymbol{\beta}}_{EGLS}$$

$$\mathbf{V\hat{a}r}(\hat{y}\,|\,\mathbf{x}_h) = \mathbf{X}_h{}'(\mathbf{V\hat{a}r}(\hat{\beta}_{EGLS}))\,\mathbf{X}_h$$

Maximum Likelihood Instead of GLS

For the linear model:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \mathrm{Var}(\boldsymbol{\varepsilon}) = \sum$$

We may be able to estimate the coefficients and the error covariance matrix at the same time, using maximum likelihood search methods. This appears to be more attractive, but, since a search method is needed, we may end up with a search in the wrong place, that really isn't the maximum likelihood solution.

Mixed Linear Models

Mixed linear models are *linear models* with a *continuous y-variable* having *fixed effects*, *random error*, and *additional random effects*:

$$\mathbf{Y} = \mathbf{X\beta} + \mathbf{Zu} + \boldsymbol{\varepsilon} \qquad \mathrm{V}(\mathbf{Y}) = \mathbf{ZGZ}' + \mathbf{R}$$

where:

- $\mathbf{X\beta}$ is the fixed part of the equation, sometimes labeled "fixed effects", assuming the *x*'s are fixed, and are not a sample from a distribution.
- The other two terms on the right hand side of the equation are random parts of the equation. **Z** is called the "design matrix", and **u** coefficients to be estimated, and these are collectively called the "random effects". Measures of these are considered to be samples from a distribution.
- The terminal error term is also random, and is called the "residual error", with error covariance matrix **R**. Measures of these are considered samples from a distribution.
- All random components have a specified probability distribution, which are all Normal distributions for mixed linear models.

<u>The Mixed Linear Model as a GLS Model</u>

Previously, we had the GLS model:

$$\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \qquad\qquad \mathrm{Var}(\boldsymbol{\varepsilon}) = \Sigma$$

The mixed linear model <u>can be viewed</u> as the same as the GLS model, where the error covariance matrix is:

$$\mathrm{Var}(\boldsymbol{\varepsilon}) = \Sigma = \mathbf{ZGZ}' + \mathbf{R}$$

and <u>the random components are be viewed as a "nuisance",</u> used only to get good estimates of standard errors for the coefficients of the fixed component of the model and of the mean of *y* given values for the *x*'s.

For GLS, using least squares:

- if we knew this matrix, we could get unbiased estimates of coefficients and variances/ covariances of coefficients.
- OR we could get a consistent estimate of the error covariance matrix and use EGLS to get unbiased estimates of coefficients, and consistent estimates of their variances/ covariances.
- Iterated EGLS could be used to obtain the maximum likelihood solution.

Examples:

1. Unequal variances of errors
2. Correlation of errors
3. Unequal variances + correlation of errors

The Mixed Linear Model as a Hierarchical Model

- Samples may be samples are from a hierarchy (e.g., trees are measured within plots sampled from a land base; samples are taken from a number of boards, etc.
- The mixed linear model can then be used to divide the error terms by levels of the hierarchy where $\mathbf{Zu}$ represents the level-2 error (e.g. plots) and $\boldsymbol{\varepsilon}$ represents the level-1 error (e.g., trees)
- This allows for different levels of prediction (population and at level-2 and level-1) and comparison of variation at different levels.

The Mixed Linear Model Where Random-Effects are of Interest

The random components may related to a random-effects factor. Examples:
- variance due to block, species or genetics where results may be from a formal experiment
- where variation is due to a hierarchy, differences in coefficients (random coefficients modelling) may be of interest, along with explanatory variables to explain these differences.

Finding a Solution
For mixed linear models we can use (e.g., PROC MIXED in SAS):
- **Maximum likelihood (ML)** to estimate all parameters (coefficients for the fixed and random effects part, and all variances covariances of R) at the same time
- OR we can use **Restricted (also called Residual) Maximum Likelihood (REML)** to obtain estimates of the random effects
  - o Use OLS to estimate the fixed part (will be unbiased estimates of the coefficients even under correlation of the error terms and/or heteroskedasticity),
  - o get the residuals from the OLS fit, which will be all random parts combined.
  - o find the estimates of the random components and variances/covariances based on a ML fit using the residuals
- Cannot compare ln*L* between a ML fitted model to ln*L* using a REML approach, since ML uses the distribution of *y* and REML uses the distribution of the residuals (all random components combined for this).
- REML is often used as the searches can more likely reach a global maximum, since the fixed parameters (the coefficients) are already estimated in the OLS fit.

The fit of mixed linear models then depends on the full specification of the model, which includes:
1. The fixed effects part of the model, including any transformations used to obtain a linear model, as with OLS and GLS.
2. The random effects part of the model, and their probability distributions.
3. The residual error distribution (including the residual error covariance matrix).

These are more complicated to understand, to set up, since you need to choose models for the fixed and random effects parts.

Because a search is needed using maximum likelihood, the solution:
1. May not converge
2. May converge but not be a global maximum
3. May converge, but the Hessian matrix could be singular.
4. The **G** and **R** matrices could be “not positive definite” – some variances could be negative, for example.

Using SAS PROC MIXED, all random components are assumed to follow normal distributions.

There are also:

1. Nonlinear mixed models (called NLMIXED in SAS), when the fixed effect is a nonlinear equation, and
2. Generalized linear mixed models (called GLIMMIX in SAS, but this is still somewhat experimental), when the random components are not normally distributed.

(See Fig. 1.3 and 1.4 taken from Littell et al. 2006. SAS for mixed models, 2nd ed. SAS, Inc., Cary, NC. for a flowchart on models and appropriate SAS procedures.)

Linear mixed models are appropriate for a wide number of problems. We will look at some that are commonly used in biological and social sciences.

Fixed and Random Components of an Experiment

In experimental design, some factors may be *fixed-effects* (all levels of interest are in the experiment) or *random-effects* (levels represent some of the levels that exist – not all levels are in the experiment). There are many variations to experimental design. For example:

1) You have a randomized block design with one factor (treatment), the blocks are often random effects (only some blocks in the experiment) and are part of the $\mathbf{Zu}$ matrix, variance due to blocks is in the **G** matrix, and the treatments can be fixed (all treatment levels of interest are in the experiment) and part of the $\mathbf{X\beta}$.

2) An experiment with fixed-effects treatments, modeled by the $\mathbf{X\beta}$ matrix, has a number of experimental units per treatment (level at which the treatment is applied), and samples within experimental units. The error covariance matrix can then be separated into the variance of experimental units that appears in the **G** matrix, but are specified in the model in the $\mathbf{Zu}$ matrix, and the variance of samples within experimental units that are in the **R** matrix.

Linear Mixed Models with One Random Component

When there is one component, the $\mathbf{Zu}$ matrix is a zero matrix and drops out of the model. This then becomes:

$$\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \qquad \mathrm{Var}(\boldsymbol{\varepsilon}) = \mathbf{R}$$

Where the matrices **Y, X**, $\boldsymbol{\beta}$, and $\boldsymbol{\varepsilon}$ are same as for the OLS (and GLS) model, and **R** is represents the error covariance matrix.

As noted earlier, the most general error covariance matrix is:

$$\mathbf{R} \text{ or } \Sigma = \begin{bmatrix} \sigma^2_{\varepsilon_1} & \sigma_{\varepsilon_1\varepsilon_2} & \sigma_{\varepsilon_1\varepsilon_3} & \cdots & \sigma_{\varepsilon_1\varepsilon_n} \\ \sigma_{\varepsilon_2\varepsilon_1} & \sigma^2_{\varepsilon_2} & \sigma_{\varepsilon_2\varepsilon_1} & \cdots & \sigma_{\varepsilon_2\varepsilon_n} \\ \sigma_{\varepsilon_3\varepsilon_1} & \sigma_{\varepsilon_3\varepsilon_2} & \sigma^2_{\varepsilon_3} & \cdots & \sigma_{\varepsilon_3\varepsilon_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{\varepsilon_n\varepsilon_1} & \sigma_{\varepsilon_n\varepsilon_2} & \sigma_{\varepsilon_n\varepsilon_3} & \cdots & \sigma^2_{\varepsilon_n} \end{bmatrix}$$

Which is called an "unstructured error covariance matrix" (TYPE=UN in SAS). This is the unconstrained error covariance matrix, with <u>*n*+(*n* X *n*)-*n*)/2 parameters to estimate</u> (The off-diagonal elements are equal as this is a symmetric matrix). It is difficult to obtain a consistent estimate of this matrix, and you may not get convergence in a ML or REML search.

Under iid this becomes the error covariance matrix for OLS.

$$\mathbf{R} = \begin{bmatrix} \sigma_\varepsilon^{\,2} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_\varepsilon^{\,2} & 0 & \cdots & 0 \\ 0 & 0 & \sigma_\varepsilon^{\,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_\varepsilon^{\,2} \end{bmatrix} = I_n \sigma_\varepsilon^{\,2}$$

an OLS should be used, since LS gives a <u>unique solution</u> (no need to search), and <u>good small sample properties.</u> This <u>is the default</u> for the **R** matrix for PROC MIXED.

Under independence but heteroskedasticity, **R** becomes the diagonal matrix for weighted LS.

$$\mathbf{R} = \begin{bmatrix} \sigma_{\varepsilon_1}{}^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{\varepsilon_2}{}^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{\varepsilon_3}{}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{\varepsilon_n}{}^2 \end{bmatrix}$$

There will be n parameters to estimate. EGLS, specifically, weighted LS can be used, by using the residuals from an OLS fit to model the variance pattern, and use 1/est. variance$_i$ as the weight. Alternatively, you can use PROC MIXED and a model for this est. variance.

Under first order autocorrelation, but homoskedasticity, this becomes the AR(1) error covariance matrix:

$$\mathbf{R} = \frac{\sigma_v{}^2}{(1-\rho)} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ 0 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix}$$

There are only two parameters to estimate. EGLS is a good choice (under PROC AUTOREG). Alternatively, you can use SUBJECT=Intercept TYPE=AR(1) in PROC MIXED to specify the error covariance matrix of AR(1) for a single subject (all data is one time series).

For more complex patterns of heteroskedasticity and/or correlation, ML or REML may be preferred over EGLS, to find estimates for the parameters of the error covariance matrix.

In biological and social datasets, this often occurs with panel/ longitudinal/ pooled cross-sectional /repeated measures data (more on these later with examples, since we will likely want to include the **Zu** for these problems).

(See SAS PROC MIXED for more error covariance matrix structures (TYPE)).

For these problems, the process for fitting using a mixed linear model is similar to EGLS:

1. Select the X variables for the fixed effects part of the linear mixed model.
2. Specify the pattern for the error covariance matrix. **NOTE:** This pattern will likely vary for different choices of the X variables matrix.
3. Find the estimated coefficients for the fixed part of the model and the variance/ covariance of the error covariance matrix:
   - at the same time using ML (method=ML in sas). May not result in a good solution (search does not converge or Hessian matrix is singular) if there are many parameters (many coefficients in the fixed part, plus many variance/covariance parameters to estimate)

   OR
   - use OLS for the fixed part and REML for the variances in a two-step fitting approach (method=REML in SAS). A final step using EGLS to find the estimated error variance/covariance matrix to find coefficients for the fixed part may be used. This reduces the number of parameters in the REML search to only those of the error covariance matrix.

Model goodness of fit:

1. ln*L* -- this can be compared among models if you are using the same y variable and within the same method of fitting. Cannot compare this between method=ML versus method=REML. Bigger is better. NOTE: Using -2 ln *L*, smaller is better.
2. Residual plots – as there is only one error as with OLS and GLS, the residual plot should be used to check whether the relationship between y and the x's is linear or whether transformations of y and/or x or a nonlinear model is needed. Look for lack of fit in the residual plot. As with OLS and GLS, this assumption must be met.
3. AIC (and others such as AICC for smaller sample sizes): smaller is better. Adds a penalty to -2 ln *L* for more parameters. NOTE: The penalty is much under debate, since this should include all parameters, coefficients of the fixed part of the model, plus all variance/covariance parameters estimated using the sample data.

Hypothesis Tests:

1. Likelihood ratio – can use this to compare nested models, for example:
   a. Model 1 uses x1, x2, x3, whereas model 2 specifies x2 and x3 only. Both specify TYPE=AR(1). Model 2 is nested in 1 (a restricted version of Model 1.
   b. Model 1 uses x1 to x3, Type=AR(1), whereas model 4 uses x1 to x3, and no type (iid errors). Model 4 is a restricted version of Model 1.
2. Wald's tests (may be asymptotic t or z tests) using Chi-square distribution for individual coefficients of the fixed part of the model.
3. Hypothesis tests for variances/covariances = 0. (can get these but still experimental).

Confidence Intervals:

Can get confidence intervals for each coefficient in the fixed part of the model, and for the predicted y given x (i.e., mean y given x), using z – may use t also in some packages.

NOTE: Degrees of freedom to be used in tests and CIs are under debate. SAS uses a default, but you can ask for others, depending upon your problem.

PROC MIXED example to fit a model with heteroskedastic error terms, instead of using weighted least squares.

Data: Volume100.xls

Model Statement: $\mathbf{X\beta}$

```
model volume=dbh dbhsq/ solution cl covb
residual outp=cond residual outpm=marg
 residual;
```

(cond are the conditional residuals and predicted values whereas marg are the marginal residuals and predicted values)

Random: $\mathbf{Zu} = \mathbf{0}$

no random effects outside of the final error term.

[conditional and marginal errors will be the same for this model]

Error Covariance Structure: **R**

Assuming observations are independent, and based on our earlier OLS fit, the errors appear to be heteroskedastic.

1. assuming iid. No “repeated” statement Save Xbeta in a SAS temporary file.
2. assuming variance is a power of the mean where mean=Xbeta obtained in the first fit.

```
repeated/local=pom(sf) r=1,2,3,98,99,100;
```

The estimated variances will be printed for the first and last three observations. Save estimated variances in a SAS temporary file, to plot wtresid = resid * 1/(est. variance^0.5) by wtpred plot, and to test normality of the weighted residuals.

```
PROC IMPORT OUT= WORK.trees
DATAFILE= "E:\FRST530\examples\mixed\volume100.xls"
DBMS=EXCEL2000 REPLACE;
   GETNAMES=YES;
RUN;

options ls=70 ps=50 pageno=1 nodate;

title1 ' ';
data trees2;
set trees;
dbhsq=dbh**2;
run;

* add a treeno to each tree in the dataset;
proc sort data=trees2;
by dbh;
run;

data treenos;
do treeno=1 to 100;
 output;
end;
run;

data trees2;
merge treenos trees2;
run;

Proc reg data=trees2;
title1 'OLS';
model volume=dbh dbhsq;
output out=pout p=yhat r=resid;
run;
proc plot data=pout;
plot resid*yhat='*';
run;

* sort by the dbhs to get a nice plot;
proc sort data=pout;
by dbh;
run;
```

```
* first plot the data and the OLS fit of a model
overlaid on top of the data;

GOPTIONS RESET=ALL;
* AXIS1 is the x axis, HORIZONTAL;
AXIS1 LENGTH=4.5 IN MINOR=NONE ORDER=0 TO 60 BY 10
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS 'dbh') ;

* AXIS2 is the y axis, VERTICAL;
AXIS2 LENGTH=4.5 IN MINOR=NONE ORDER=0 TO 2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0 F=SWISS
'volume') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.3 CM;

SYMBOL1 W=1 C=BLACK L=1 v=star   ;
SYMBOL2 W=1 C=BLACK L=2 v=circle  ;
SYMBOL3 W=1 C=BLACK L=3 v=square  ;
SYMBOL4 W=1 C=BLACK L=3 v=none i=join ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C
 'Volume versus dbh' ;
TITLE2 ' ' ;

PROC GPLOT DATA = pout;
        PLOT volume*dbh=1 yhat*dbh=4/overlay
        VAXIS = AXIS2
        HAXIS = AXIS1;
RUN ;
proc sort data=trees2;
by dbh;
run;

* repeat the fit of the linear model using PROC MIXED, with iid
error structure.  save the solution, xb, in an output file using
ODS to be used in the next step to estimate variances of R;
ods output SolutionF=sf ;
proc mixed data=trees2 method=ml;
title1 'MIXED MODEL 1: only one error term, as in OLS';
 model volume=dbh dbhsq/solution cl covb
 residual outp=cond residual outpm=marg residual;
run;
* since there is no Z matrix,marginals and conditional predicted
and residual values
are the same;
proc plot data=marg;
    plot resid*pred='*';
run;
proc univariate data=marg plot normal;
           var resid;
run;

* repeat the fit of the linear model using PROC MIXED, with non
iid error structure, specifically variances are powers of the xb
matrix. List first and last 3 estimated error variances. Save the
solution in order to get weighted residual plots;
ods output CovParms=covparms;
proc mixed data=trees2 method=ml covtest;
```

```
title1 'MIXED MODEL 2: error variances as a function of xb';
 model volume=dbh dbhsq/ solution cl covb
 residual outp=cond2 residual outpm=marg2 residual;
 repeated/local=pom(sf) r=1,2,3,98,99,100;
run;

* since there is no Z matrix,marginals and  conditional predicted
and residual values are the same. Must weight the residuals and
predicted values using sqrt (1/est. variance) to check if the
variance model worked. First isolate the estimated power (POM)
needed for est. variances;
data pom;
set covparms;
if covparm='POM' then pom=estimate;
if covparm ne 'POM' then delete;
do treeno=1 to 100;
  pom=pom;
  output;
  end;
keep treeno pom;
run;

data marg2w;
merge pom marg2;
by treeno;
estvar=pred**pom;
rtwt=1/(estvar**0.5);
wtpred=rtwt*pred;
wtresid=rtwt*resid;
run;

proc plot data=marg2w;
  plot resid*pred='*';
  plot wtresid*wtpred='*';
run;
proc univariate data=marg2w plot normal;
  var wtresid;
run;
```

The REG Procedure
Model: MODEL1
Dependent Variable: volume volume

Number of Observations Read	100
Number of Observations Used	100

Analysis of Variance

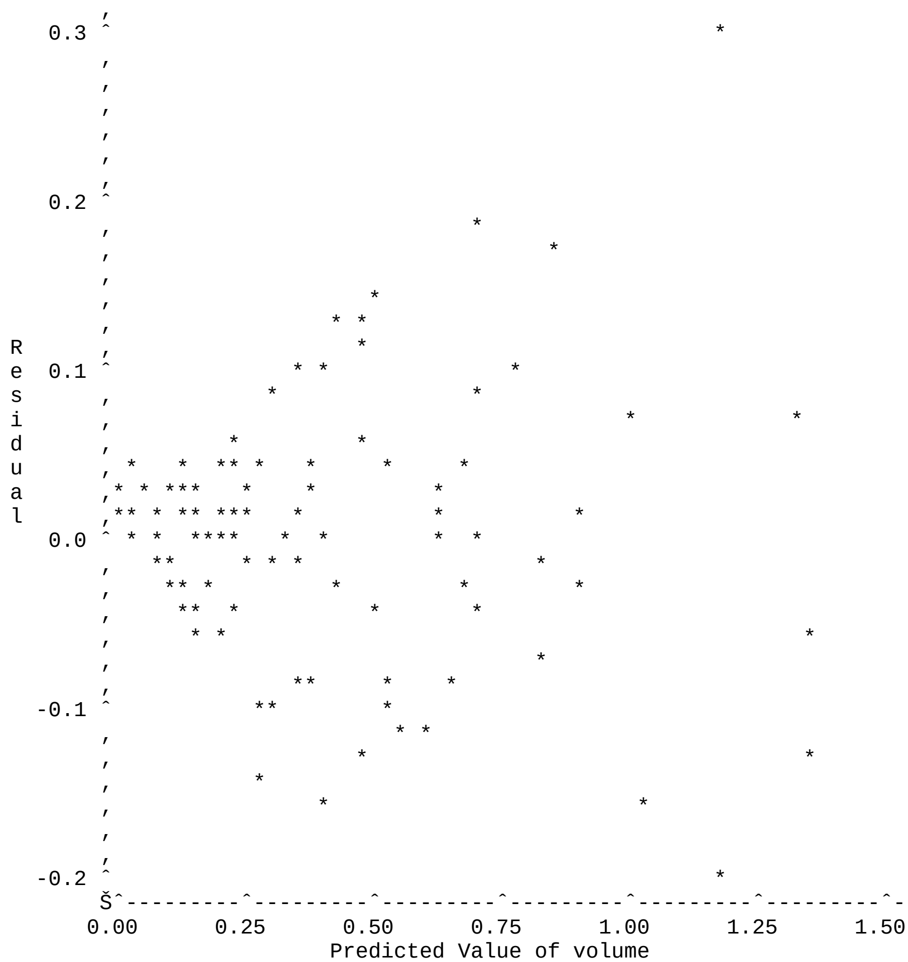
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	9.95249	4.97624	846.89	<.0001
Error	97	0.56996	0.00588		
Corrected Total	99	10.52245			

Root MSE	0.07665	R-Square	0.9458
Dependent Mean	0.39446	Adj R-Sq	0.9447
Coeff Var	19.43248		

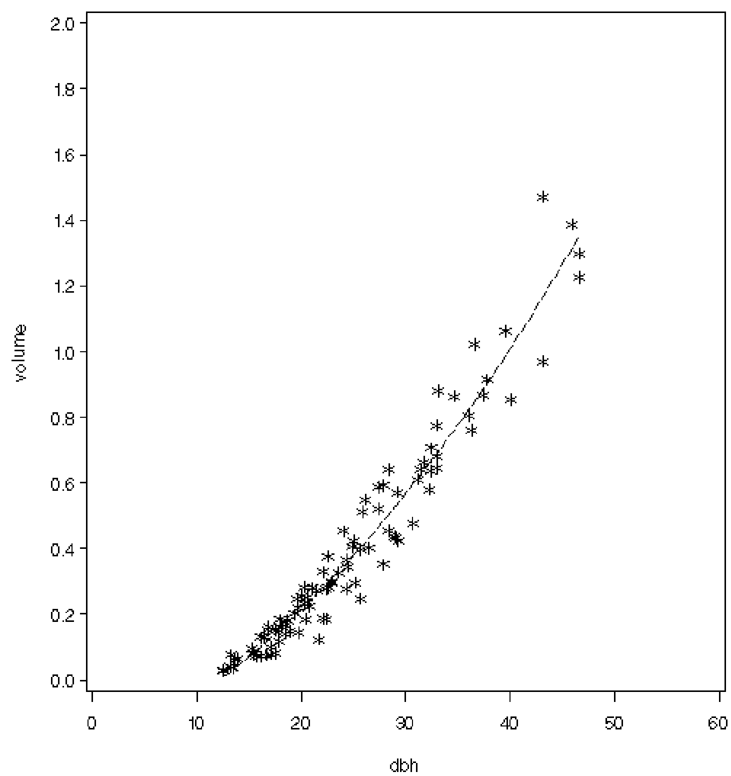
Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	Intercept	1	-0.22125	0.06954	-3.18	0.0020
dbh	dbh	1	0.01289	0.00532	2.42	0.0172
dbhsq		1	0.00044628	0.00009414	4.74	<.0001

Plot of resid*yhat. Symbol used is '*'.



Volume versus dbh



The Mixed Procedure

Model Information

Data Set	WORK.TREES2
Dependent Variable	volume
Covariance Structure	**Diagonal**
Estimation Method	**ML**
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Residual

Dimensions

Covariance Parameters	**1**
Columns in X	**3**
Columns in Z	**0**
Subjects	**1**
Max Obs Per Subject	**100**

Number of Observations

Number of Observations Read	100
Number of Observations Used	100
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate
Residual	0.005700

Fit Statistics

-2 Log Likelihood	**-232.9**
AIC (smaller is better)	-224.9
AICC (smaller is better)	-224.5
BIC (smaller is better)	-214.5

The Mixed Procedure

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	-0.2212	0.06849	97	-3.23	0.0017	0.05
dbh	0.01289	0.005237	97	2.46	0.0156	0.05
dbhsq	0.000446	0.000093	97	4.81	<.0001	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-0.3572	-0.08531
dbh	0.002493	0.02328
dbhsq	0.000262	0.000630

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3
1	Intercept	0.004691	-0.00035	5.964E-6
2	dbh	-0.00035	0.000027	-4.78E-7
3	dbhsq	5.964E-6	-4.78E-7	8.597E-9

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
dbh	1	97	6.06	0.0156
dbhsq	1	97	23.17	<.0001

(Residual plot same as for OLS, since there is only one error term and marginal=conditional residuals)

The UNIVARIATE Procedure
Variable: Resid (Residual)

Normal Probability Plot

```
 0.31+                                                  *
     |
     |
     |
     |
 0.21+
     |                                              *  ++
     |                                            *  ++
     |                                            +++
     |                                         ***
 0.11+                                      ***
     |                                    +*+
     |                                 +++**
     |                               ++****
     |                            ++****
 0.01+                         *****
     |                     ****
     |                  ***++
     |                ***+
     |              **++
-0.09+            ***
     |          **
     |       +**
     |    *+**
     |  ++
-0.19+*+
      +----+----+----+----+----+----+----+----+----+----+
          -2        -1         0        +1        +2
```

The Mixed Procedure

Model Information

Data Set	WORK.TREES2
Dependent Variable	volume
Covariance Structure	**Power of the Mean**
Estimation Method	**ML**
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Between-Within

Dimensions

Covariance Parameters	**2**
Columns in X	**3**
Columns in Z	**0**
Subjects	**100**
Max Obs Per Subject	**1**

Number of Observations

Number of Observations Read	100
Number of Observations Used	100
Number of Observations Not Used	0

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	-232.94800466	
1	2	-288.09431305	0.04496979
2	1	-299.44327829	0.00069676
3	1	-299.61445651	0.00000047
4	1	-299.61456897	0.00000000

Convergence criteria met.

The Mixed Procedure

Estimated R Matrix for Index 1

Row	Col1
1	0.000025

Estimated R Matrix for Index 2

Row	Col1
1	0.000034

Estimated R Matrix for Index 3

Row	Col1
1	0.000108

Estimated R
Matrix for

Index 98

Row	Col1
1	0.02841

Estimated R
Matrix for
Index 99

Row	Col1
1	0.02959

Estimated R
Matrix for
Index 100

Row	Col1
1	0.02959

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z
POM	**1.4292**	**0.1459**	**9.80**	**<.0001**
Residual	**0.01919**	**0.004576**	**4.19**	**<.0001**

Fit Statistics

-2 Log Likelihood	**-299.6**
AIC (smaller is better)	-289.6
AICC (smaller is better)	-289.0
BIC (smaller is better)	-276.6

Null Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
1	66.67	<.0001

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	-0.1101	0.03120	97	-3.53	0.0006	0.05
dbh	0.003423	0.003320	97	1.03	0.3051	0.05
dbhsq	0.000627	0.000079	97	7.94	<.0001	0.05

The Mixed Procedure

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-0.1721	-0.04823
dbh	-0.00317	0.01001
dbhsq	0.000470	0.000784

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3
1	Intercept	0.000973	-0.00010	2.326E-6
2	dbh	-0.00010	0.000011	-2.56E-7
3	dbhsq	2.326E-6	-2.56E-7	6.231E-9

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
dbh	1	97	1.06	0.3051
dbhsq	1	97	63.08	<.0001

MIXED MODEL 2: error variances as a function of xb   14

Plot of Resid*Pred. Symbol used is '*'.

```
        ,
        ,
    0.3 ^
        ,
        ,                                                *
        ,
        ,
        ,
    0.2 ^
        ,                            *
        ,                                  *
        ,                    *
        ,                *  *
        ,             *  *  *
    0.1 ^                               *
        ,           *               *
 R      ,         *        *
 e      ,        *  *   *     *     *            *
 s      ,  *  **  **              *
 i      ,  * *** ***  **          *                             *
 d  0.0 ^ ** * ** * ** * *       *  *         *
 u      ,  * *            *        *
 a      ,    **** *          *      *     *  *
 l      ,     ** *
        ,              *           *
        ,           *   *     *            *
   -0.1 ^           *         * *
        ,                    *                                     *
        ,          *
        ,                *
        ,
        ,                                         *               *
   -0.2 ^
        ,
        ,                                                *
        ,
        ,
        ,
   -0.3 ^
        ,
        Š^---------^---------^---------^---------^---------^---------^-
       0.00      0.25      0.50      0.75      1.00      1.25      1.50

                                 Predicted Mean
```

```
NOTE: 18 obs hidden.

         MIXED MODEL 2: error variances as a function of xb       15

             Plot of wtresid*wtpred.  Symbol used is '*'.

wtresid ,
        ,
    0.3 ^
        ,             *
        ,                                *  *
        ,                                    **    *        *
        ,                              *   *
    0.2 ^                           *        *
        ,                                             *
        ,                     *    *
        ,                           * *
        ,                     *              *     * *
    0.1 ^                    *  *    *    *
        ,                           *     *    *
        ,              **     *    *  *           *      *
        ,                  *   *    *    *
        ,                       ***     *  *     *             *
    0.0 ^          *             *     ***       * *   *
        ,           * *         *    *                *
        ,             *    *                *     *    *
        ,                                     *    *
        ,                   *                         *
   -0.1 ^                        ** *             *            *
        ,              *    *                  *
        ,                      **        *     *               *
        ,                                 *    * *
        ,                    *     *                      *
   -0.2 ^                     *              *              *
        ,                      *      *
        ,                      *       *
        ,
        ,                                  *
   -0.3 ^
        ,
        ,
        ,                             *
        ,
   -0.4 ^
        ,
        Š^-----------^-----------^-----------^-----------^-----------^
        0.2         0.4         0.6         0.8         1.0        1.2

                                    wtpred

NOTE: 5 obs hidden.
```

```
         MIXED MODEL 2: error variances as a function of xb        16

                      The UNIVARIATE Procedure
                         Variable:  wtresid

                               Moments

 N                         100    Sum Weights                100
 Mean               -0.0006412    Sum Observations    -0.0641239
 Std Deviation      0.13684252    Variance            0.01872587
 Skewness           -0.0229452    Kurtosis            -0.4272948
 Uncorrected SS     1.85390268    Corrected SS        1.85386156
 Coeff Variation    -21340.335    Std Error Mean      0.01368425
```

(Some outputs removed)

```
                         Tests for Normality

    Test                  --Statistic---    -----p Value------

    Shapiro-Wilk          W     0.986858    Pr < W      0.4281
    Kolmogorov-Smirnov    D     0.051645    Pr > D     >0.1500
    Cramer-von Mises      W-Sq   0.04624    Pr > W-Sq  >0.2500
    Anderson-Darling      A-Sq  0.329901    Pr > A-Sq  >0.2500

        MIXED MODEL 2: error variances as a function of xb        18

                      The UNIVARIATE Procedure
                         Variable:  wtresid

                         Normal Probability Plot
      0.275+                                          *+*   *
           |                                    ******
           |                                  **++
           |                               ****
           |                           ****
           |                       *****
           |                   *****
           |                +**+
           |             ****
           |         +***
           |    ******
           |  *++
           |++++
     -0.375+*
            +----+----+----+----+----+----+----+----+----+----+
                -2        -1         0        +1        +2
```

Hierarchical Models: Separation of Errors
(also called clustered data)

Samples come from different physical scales:

- For example, we locate a plot of trees (level 2; a cluster in space) and then measure trees in the plot (level 1). (i.e., multistage sampling)
- y is measured at level 1, the lowest level in the hierarchy.
- With two levels, level 1 is the *individual* and level 2 is the *subject*, called a *two-level hierarchical model*.
- There can be more than 2 levels in the hierarchy.
- Predictor variables, x, can be at all levels.

We can used mixed models (PROC MIXED in SAS):

1. to estimate the coefficients for the fixed component of the model,
2. to estimate variances/covariances for the different levels,
3. to estimate the effects of level-2 for each observation in the sample, and
4. to estimate predicted values at different levels in the hierarchy, along with associated confidence intervals

The error variances/ covariances for level 1 are found in the **R** matrix.
The error variances /covariances for level 2 are found in the **G** matrix.

For each observation, using i for tree (individual) and j for plot (subject) and including only one explanatory variable and only two plots (level-2), the model is:

$$\begin{aligned} y_{ij} &= \beta_0 + \beta_1 x_{1ij} + u_1 z_{1ij} + u_2 z_{2ij} + \varepsilon_{ij} \\ &= X_{ij}\beta + Z_j u + \varepsilon_{ij} \end{aligned}$$

$$z_{1ij} \begin{cases} 1 \text{ for subject } 1 \\ 0 \text{ for subject } 2 \end{cases}$$

$$z_{2ij} \begin{cases} 0 \text{ for subject } 1 \\ 1 \text{ for subject } 2 \end{cases}$$

- The intercept of the model is the population-averaged intercept.
- The design matrix, **Z**, in this case is just dummy variables that alter the intercept for different plots (*random intercept*).
- The values for all individuals within a particular subject will all be the same (same set of dummy variables).
- This can be alternatively written as:

$$y_{ij} = \beta_0 + b_{0j} + \beta_1 x_{1ij} + \varepsilon_{ij}$$

where b_{0j} is the change in the intercept for subject j.

- We will get estimates of (i) the intercepts for the population, (ii) for each plot in the sample (not often of interest), AND (ii) estimates of the variances due to plots and due to trees.

Comparing this View of a Hierarchical Model, to OLS Using the Subject as a Class Variable

We could use subject (plot in this case), as an explanatory variable (a set of dummy variables, where the number of dummy variables = # subjects – 1) to alter the intercept for different subjects.

The model would look similar except that

- There would be no population-averaged intercept. Instead, this first intercept would represent the intercept for the subject where all dummy variables are equal to zero. Therefore, we would have no estimate of the population-averaged intercept.
- With OLS using plots as class variables, we would get estimates of all the subject intercepts for those subjects in the sample, but these may not be of interest.
- We would not get an estimate of the variance due to plots versus trees, since including the plot (the subject) as a class variable in OLS assumes that all plots of interest are in the sample – that plot is a fixed-effect, and there is no distribution.

Viewing This as a Complex Error Covariance Matrix Structure:

Another way of looking at this mixed model:

$$y_{ij} = \beta_0 + \beta_1 x_{1ij} + \varepsilon_j + \varepsilon_{ij}$$

Where each individual has two sources of error,

1. The difference between the subject mean over all individuals and the population mean over all individuals (subject level error), and
2. the difference between the measurement for individual *i* in subject *j*, and the mean for subject *j*.

The combined errors will have an error covariance matrix that has a pattern called *compound symmetry*. (example of matrices to follow).

Comparison of this View of a Hierarchical Model to Sampling and to Experimental Design:

- This would be the between error in analysis of covariance, versus the within error.
- In sampling, this would be the first stage variance versus the second stage variance using multistage regression sampling.
- This is sometimes used to model the two sources of error in split-plot experiments, where the “whole-plot” is the subject and the “split-plot” is the individual.

Matrices Using Two Subjects as an Example:

$\hat{\mathbf{Y}}$ (*n* X 1), $\mathbf{Y}$ (*n* X 1), $\mathbf{X\beta}$ (*n* X *p*) (*p* X 1), and $\boldsymbol{\varepsilon}$ (*n* X 1) appear as they do with OLS, with first column of the **X** matrix as a column of 1’s to represent the intercept of the linear model, and the observations for each subject in a group and stacked. **R** appears as shown for models without other random factors (i.e., as OLS when iid or GLS when not iid). How do $\mathbf{Z}$ and $\mathbf{u}$ appear for this problem?

Assuming we have 2 plots, heights of 3 trees are measured in each plot:

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \varepsilon_{32} \end{bmatrix}$$

What does the error covariance matrix look like (combination of level-2 and level-1 covariance matrices)? This will be a n X n matrix.

$$\mathbf{V(Y)} = \mathbf{ZGZ'} + \mathbf{R}$$

First, what does **G** look like? This is a matrix of showing the variance of each plot mean from the grand mean, and the covariance among plots. Assuming plots are independent (covariance =0) and using τ_{00} to represent the plot variance (plot mean versus overall mean):

$$\mathbf{G} = \begin{bmatrix} \tau_{00} & 0 \\ 0 & \tau_{00} \end{bmatrix}$$

Then:

$$\mathbf{Z\ G} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} \tau_{00} & 0 \\ 0 & \tau_{00} \end{bmatrix} = \begin{bmatrix} \tau_{00} & 0 \\ \tau_{00} & 0 \\ \tau_{00} & 0 \\ 0 & \tau_{00} \\ 0 & \tau_{00} \\ 0 & \tau_{00} \end{bmatrix}$$

$$\mathbf{ZGZ'} = \begin{bmatrix} \tau_{00} & 0 \\ \tau_{00} & 0 \\ \tau_{00} & 0 \\ 0 & \tau_{00} \\ 0 & \tau_{00} \\ 0 & \tau_{00} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0 \\ \tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0 \\ \tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00} \\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00} \\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00} \end{bmatrix}$$

$$\mathbf{ZG\,Z'}+\mathbf{R}=\begin{bmatrix}\tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0\\ \tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0\\ \tau_{00} & \tau_{00} & \tau_{00} & 0 & 0 & 0\\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00}\\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00}\\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00}\end{bmatrix}$$

$$+\begin{bmatrix}\sigma_\varepsilon^2 & 0 & 0 & 0 & 0 & 0\\ 0 & \sigma_\varepsilon^2 & 0 & 0 & 0 & 0\\ 0 & 0 & \sigma_\varepsilon^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \sigma_\varepsilon^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \sigma_\varepsilon^2 & 0\\ 0 & 0 & 0 & 0 & 0 & \sigma_\varepsilon^2\end{bmatrix}$$

$$=\begin{bmatrix}\tau_{00}+\sigma_\varepsilon^2 & \tau_{00} & \tau_{00} & 0 & 0 & 0\\ \tau_{00} & \tau_{00}+\sigma_\varepsilon^2 & \tau_{00} & 0 & 0 & 0\\ \tau_{00} & \tau_{00} & \tau_{00}+\sigma_\varepsilon^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \tau_{00}+\sigma_\varepsilon^2 & \tau_{00} & \tau_{00}\\ 0 & 0 & 0 & \tau_{00} & \tau_{00}+\sigma_\varepsilon^2 & \tau_{00}\\ 0 & 0 & 0 & \tau_{00} & \tau_{00} & \tau_{00}+\sigma_\varepsilon^2\end{bmatrix}$$

This is a *block diagonal matrix*, with elements outside of the blocks all being zero, indicating there is no correlation between trees of different plots. Each block shows the error covariance matrix for a plot (level-2). The error variance for each tree is the sum of variance of plot values (means of trees) around the overall mean, and variance of tree values around plot means. The trees within a plot are related, and the covariance is equal to the plot variance. The errors can therefore be modeled as (i) two parts, specifying G and R, using a "random intercept"; or as (ii) one part, using this entire matrix with TYPE=CS for compound symmetry.

Option (i) is better in that we can get estimates of a plot predictions (means) at the intermediate scale, and population-averaged means (broad scale), and estimates of plot and trees within plots variances to examine the sources of residual variation.

Prediction at Broad, Intermediate and Narrow Scales

Broad Scale

- Predicted values are the marginal means where $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$, using only the fixed effects to get estimates of the *mean of y* given values for the *x*'s. (Same as with OLS).
- These are often called *population averaged* or "at the universe scale", and are called the *marginal means* in SAS
- The associated residual is calculated as the difference between $\mathbf{Y}$ and $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$, and is a sum of level-1+level-2 errors, called the *marginal error* in SAS.

Intermediate Scale

- Using the example where plots are at level-2, the estimates for each plot are found by:

$$\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$$

- These are called *conditional means* in SAS, or the *subject-level* means, representing an estimate of the average of trees within a plot,
- The residuals as just the terminal error called *conditional error* in SAS (often called "white noise" in textbooks), which would be the difference between the tree measures within the plot, and the plot average for that plot.

Narrow Scale: For this example, this would be the actual measures for each tree.

PROC MIXED example:

Data: Three plots (q=3) with different numbers of trees in each plot. Height and age are measured on each tree (not repeated measures as only one height /age measure on each tree). A model form was selected using OLS to determine what transformations are needed to "linearize" the relationship between height and age. Several models were run to illustrate mixed models results. The number of observations is 34.

OLS:

PROC REG ...;
model height=age lnage/. . .;
X has 3 columns: intercept age lnage

$\boldsymbol{\beta}$ has 3 values.

Residuals will be plot + tree errors combined and are considered to be normal, iid.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$

The **R** matrix is iid with only one parameter, MSE estimating the variance of the combined errors.
There is no **G** matrix with OLS.

Mixed Model 1: M1
PROC MIXED ...;
model height=age lnage/solution. . .;
(no random nor repeated statements)
X has 3 columns: intercept age lnage
$\boldsymbol{\beta}$ has 3 values.
No **Z** matrix.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$ (conditional and marginal residuals are the same)
Residuals will be plot + tree errors combined (conditional and marginal residuals are the same) and are considered to be normal, iid
The **R** matrix is iid with only one parameter, the estimated variance of the combined errors.
There is no **G** matrix.

Therefore, this is only one covariance parameter (CovParms), and this represented the diagonal values on the n X n **R** matrix.

Mixed Model 2: M2
PROC MIXED ...;
model height=/solution. . .;
random intercept/subject=plot solution g;
X has 1 columns: intercept
$\boldsymbol{\beta}$ has 1 value (called *solutionf*)
Z matrix will have three columns: dummy variables for each plot
u matrix will have three values, one for each plot to adjust the population averaged intercept (called *solutiong*)
Error will be separated into plot + tree errors combined, each considered to be normal, iid (plots are independent, trees within plots are independent and have equal variance)

Marginal Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$. Will be all the same, since **X** only includes the intercept, for M2.
Marginal Residuals is the combination of plot+tree error, representing the difference between the measured tree height, and the intercept in this model (regardless of plot)
Conditional Predicted values will be the fixed part of the equation plus added change in intercept for the plot (estimates of plot level predicted heights), $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$. For this model, will be equal to the varying intercept.
Marginal Residuals is tree errors only, representing the difference between the measured tree height, and the plot estimated average height.
The **R** matrix is iid with only one parameter, the estimated variance of the tree heights within a plot, versus the plot level mean.
The **G** matrix has one value, the variance between plot level means and overall means (plot level variances).

Therefore, there will be two covariance parameters (called *CovParms*), representing the diagonal values on the n X n **R** matrix, and all elements of the q X q **G** matrix.

Mixed Model 3: M3
PROC MIXED ...;
model height=age lnage/solution. . .;
random intercept/subject=plot solution g;
X has 3 columns: intercept

$\boldsymbol{\beta}$ has 3 value (called *solutionf*)

Z matrix will have three columns: dummy variables for each plot
u matrix will have three values, one for each plot to adjust the population averaged intercept (called *solutiong*)
Error will be separated into plot + tree errors combined, each considered to be normal, iid (plots are independent, trees within plots are independent and have equal variance)

Marginal Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$. Will be the same for each observation in the sample having the same age values. Same as OLS and M1.
Marginal Residuals as with OLS and M1, is the combination of plot+tree error, representing the difference between the measured tree height, and that estimated for a particular age (regardless of plot)
Conditional Predicted values will be the fixed part of the equation plus added change in intercept for the plot (estimates of plot level predicted heights), $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Zu}$. Will be the same for all trees in a plot having the same age value.
Marginal Residuals is tree errors only, representing the difference between the measured tree height, and the plot estimated average height.
The **R** matrix is iid with only one parameter, the estimated variance of the tree heights within a plot, versus the plot level mean.
The **G** matrix has one value, the variance between plot level means and overall means (plot level variances).

Therefore, there will be two covariance parameters (called *CovParms*), representing the diagonal values on the n X n **R** matrix, and all elements of the q X q **G** matrix.

Mixed Model 4: M4
PROC MIXED ...;
model height=age lnage/solution. . .;
repeated/type=cs subject=plot r=1,2,3;
X has 3 columns: intercept
$\boldsymbol{\beta}$ has 3 value (called *solutionf*)
No **Z** matrix.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$ (conditional and marginal residuals are the same)
Residuals will be plot + tree errors combined (conditional and marginal residuals are the same) and are considered have a compound symmetry error variance.
The **R** matrix is compound symmetry with two parameters, the plot and trees within plots variances.
There is no **G** matrix.

Therefore, there are two covariance parameters *(CovParms*), used to create the block diagonal n X n **R** matrix with a compound symmetry pattern.

SAS Code:

```
PROC IMPORT OUT= WORK.trees
DATAFILE=E:\FRST530\examples\mixed\htage_plots.xls"
DBMS=EXCEL2000 REPLACE;   GETNAMES=YES;
RUN;

options ls=70 ps=50 pageno=1 nodate;

title1 ' ';
data trees2;
set trees;
agesq=age**2;
lnage=log(age);
run;

* print out the data as this is quite small;
proc print data=trees2;
run;
Proc reg data=trees2;
title1 'OLS';
model height=age lnage;
output out=pout2 p=yhat r=resid;
run;
* sort by the ages to get a nice plot;
proc sort data=pout2;
by age;
run;

* first plot the data and the OLS fit of a model
overlaid on top of the data;
GOPTIONS RESET=ALL;
* AXIS1 is the x axis, HORIZONTAL;
AXIS1 LENGTH=4.5 IN MINOR=NONE ORDER=0 TO 160 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS 'AGE') ;
* AXIS2 is the y axis, VERTICAL;
AXIS2 LENGTH=4.5 IN MINOR=NONE ORDER=10 TO 35 BY 5
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0 F=SWISS
'Height') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.3 CM;

SYMBOL1 W=1 C=BLACK L=1 v=star   ;
SYMBOL2 W=1 C=BLACK L=2 v=circle  ;
SYMBOL3 W=1 C=BLACK L=3 v=square  ;
SYMBOL4 W=1 C=BLACK L=3 v=none i=join ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C
 'Height Versus Age for each Plot' ;
TITLE2 ' ' ;

PROC GPLOT DATA = pout2;
  PLOT height*AGE=1 yhat*age=4/overlay
```

```
  VAXIS = AXIS2
  HAXIS = AXIS1;
RUN ;

*sort back by plot and tree;
proc sort data=trees2;
by plot tree;
run;
```

*** repeat the fit of the linear model using PROC MIXED;**

```
proc mixed data=trees2 method=ml;
title1 'MIXED MODEL 1: only one error term, as in OLS';
 class plot;
 model height = Age lnage / solution cl covb
 residual outp=cond residual outpm=marg residual;
run;
* use the conditional means out of MIXED first in plotting
(intermediate or wide) -- only the last error term as the
residual;
proc print data=cond;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
run;
proc plot data=cond;
 plot resid*pred='*';
run;
proc univariate data=cond plot normal;
 var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)-- all random components in
residual;
proc print data=marg;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
run;
proc plot data=marg;
  plot resid*pred='*';
run;
proc univariate data=marg plot normal;
  var resid;
run;
```

```
* fit an intercept only model using PROC MIXED, and
including the plot as a random effect, to separate out from
the error term;
proc mixed data=trees2 method=ml;
title1 'MIXED MODEL 2:split error into plot (level-2) and
tree (level-1)';
 class plot;
 model height = / solution cl covb
 residual outp=cond2 residual outpm=marg2 residual;
       random intercept/ subject=plot solution g;
run;

* use the conditional means out of MIXED first in plotting
(intermediate or narrow) -- only the last error term as the
residual;
proc print data=cond2;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
run;
proc plot data=cond2;
  plot resid*pred='*';
run;
proc univariate data=cond2 plot normal;
  var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)-- all random components in
residual;
proc print data=marg2;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
run;
proc plot data=marg2;
    plot resid*pred='*';
run;
proc univariate data=marg2 plot normal;
          var resid;
run;

* fit the linear model using MIXED, but allow for the plot
as a random effect, to separate out from the error term;
proc mixed data=trees2 method=ml;
title1 'MIXED MODEL 3: fit linear model split error into
plot (level-2) and tree (level-1)';
 class plot;
 model height = age lnage/ solution cl covb
 residual outp=cond3 residual outpm=marg3 residual;
       random intercept/ subject=plot solution g;
run;
```

```
* use the conditional means out of MIXED first in plotting
(intermediate or narrow)-- only the last error term as the
residual;
proc print data=cond3;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
run;
proc plot data=cond3;
       plot resid*pred='*';
run;
proc univariate data=cond3 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)-- all random components in
residual;
proc print data=marg3;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
run;
proc plot data=marg3;
    plot resid*pred='*';
run;
proc univariate data=marg3 plot normal;
          var resid;
run;
```

*** fit the linear model using MIXED, split the error term into plot and tree using a compound symmetry specification for R, instead of using random intercept for G and iid for R;**

```
proc mixed data=trees2 method=ml;
title1 'MODEL 4: linear model, split error into plot (level-
2) and tree (level-1)';
title2 'and allow for compound symmetry error matrix-- same
as varying intercept';
 class plot;
 model height = age lnage/ solution cl covb
 residual outp=cond4 residual outpm=marg4 residual;
       repeated/type=cs subject=plot r=1,2,3;
run;
* use the conditional means out of MIXED first in plotting
(intermediate or narrow)-- only the last error term as the
residual;
proc print data=cond4;
title3 'conditional residuals (only white noise) and
predicted values';
title4 'using fixed+random effects';
run;
proc plot data=cond4;
       plot resid*pred='*';
run;
```

```
proc univariate data=cond4 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)-- all random components in
residual;
proc print data=marg4;
title3 'marginal residuals (all random components)';
title4 'and predicted values using fixed effects only';
run;
proc plot data=marg4;
    plot resid*pred='*';
run;
proc univariate data=marg4 plot normal;
          var resid;
run;
```

Obs	plot	Tree	age	height	agesq	lnage
1	1	1	85	30.48	7225	4.44265
2	1	2	57	26.40	3249	4.04305
3	1	3	140	33.24	19600	4.94164
4	1	4	125	30.60	15625	4.82831
5	1	5	135	34.68	18225	4.90527
6	1	6	100	29.28	10000	4.60517
7	1	7	124	34.80	15376	4.82028
8	1	8	152	29.76	23104	5.02388
9	1	9	75	19.08	5625	4.31749
10	1	10	82	29.16	6724	4.40672
11	2	1	103	25.20	10609	4.63473
12	2	2	92	22.20	8464	4.52179
13	2	3	93	24.10	8649	4.53260
14	2	4	105	21.70	11025	4.65396
15	2	5	99	25.00	9801	4.59512
16	2	6	107	26.50	11449	4.67283
17	2	7	104	24.90	10816	4.64439
18	2	8	51	15.90	2601	3.93183
19	2	9	102	27.40	10404	4.62497
20	2	10	126	23.10	15876	4.83628
21	2	11	130	24.80	16900	4.86753
22	2	12	108	21.70	11664	4.68213
23	2	13	118	26.00	13924	4.77068
24	3	1	138	21.07	19044	4.92725
25	3	2	98	17.01	9604	4.58497
26	3	3	116	19.32	13456	4.75359
27	3	4	125	16.31	15625	4.82831
28	3	5	113	17.36	12769	4.72739
29	3	6	94	16.17	8836	4.54329
30	3	7	77	13.93	5929	4.34381
31	3	8	109	18.83	11881	4.69135
32	3	9	136	19.04	18496	4.91265
33	3	10	93	17.64	8649	4.53260
34	3	11	151	18.41	22801	5.01728

The REG Procedure
Model: MODEL1
Dependent Variable: height height

Number of Observations Read	34
Number of Observations Used	34

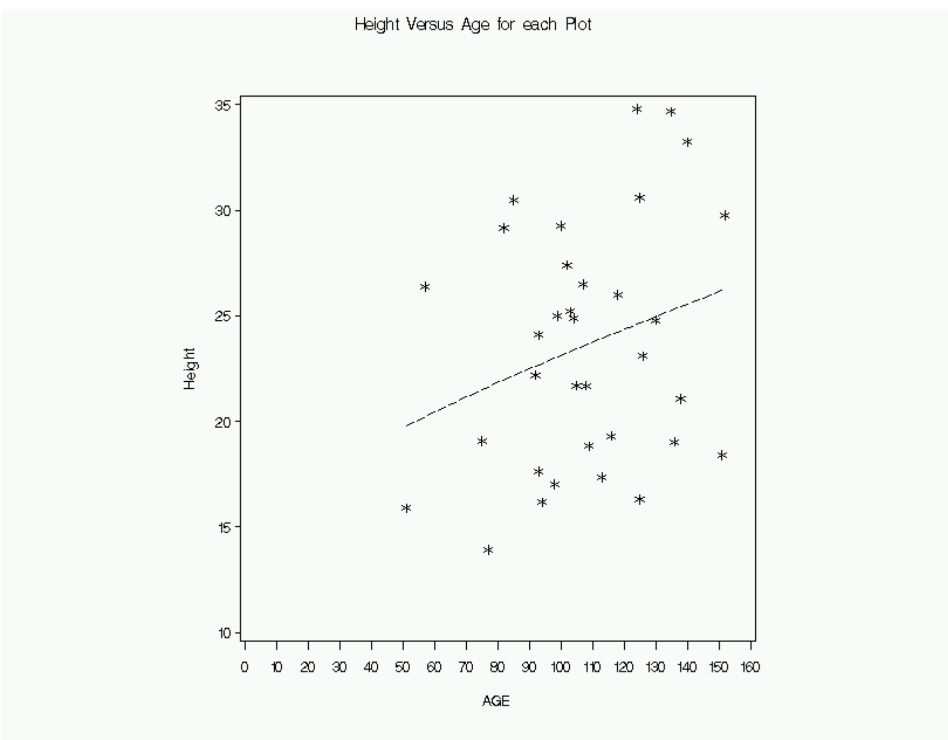
Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	77.16489	38.58244	1.18	0.3194
Error	31	1009.82259	32.57492		
Corrected Total	33	1086.98747			

Root MSE	5.70744	R-Square	0.0710
Dependent Mean	23.56088	Adj R-Sq	0.0111
Coeff Var	24.22424		

Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value	Pr>|t|
Intercept	Intercept	1	11.63441	83.70231	0.14	0.8904
age	age	1	0.04848	0.24029	0.20	0.8414
lnage		1	1.44090	23.45013	0.06	0.9514



The Mixed Procedure

Model Information

Data Set	WORK.TREES2
Dependent Variable	height
Covariance Structure	Diagonal
Estimation Method	**ML**
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	**Residual**

Class Level Information

Class	Levels	Values
plot	3	1 2 3

Dimensions

Covariance Parameters	1
Columns in X	3
Columns in Z	0
Subjects	1
Max Obs Per Subject	34

Number of Observations

Number of Observations Read	34
Number of Observations Used	34
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate
Residual	29.7007

Fit Statistics

-2 Log Likelihood	211.8
AIC (smaller is better)	219.8

The Mixed Procedure

Fit Statistics

AICC (smaller is better)	221.2
BIC (smaller is better)	225.9

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	11.6344	79.9243	31	0.15	0.8852	0.05
age	0.04848	0.2294	31	0.21	0.8340	0.05
lnage	1.4409	22.3917	31	0.06	0.9491	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-151.37	174.64
age	-0.4195	0.5164
lnage	-44.2272	47.1090

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3
1	Intercept	6387.89	17.8781	-1787.02
2	age	17.8781	0.05265	-5.0624
3	lnage	-1787.02	-5.0624	501.39

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
age	1	31	0.04	0.8340
lnage	1	31	0.00	0.9491

(only first 13 of 34 observations shown for con residuals and predicted values – missing some plot 2 and plot 3 in the list)

MIXED MODEL 1: only one error term, as in OLS
conditional residuals (only white noise) and predicted values using fixed+random effects

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	22.1570	1.36793
2	1	2	57	26.40	3249	4.04305	20.2237	3.08176
3	1	3	140	33.24	19600	4.94164	25.5427	1.76638
4	1	4	125	30.60	15625	4.82831	24.6521	1.15422
5	1	5	135	34.68	18225	4.90527	25.2478	1.50121
6	1	6	100	29.28	10000	4.60517	23.1185	1.20728
7	1	7	124	34.80	15376	4.82028	24.5920	1.13458
8	1	8	152	29.76	23104	5.02388	26.2430	2.58724
9	1	9	75	19.08	5625	4.31749	21.4918	1.58829
10	1	10	82	29.16	6724	4.40672	21.9598	1.41254
11	2	1	103	25.20	10609	4.63473	23.3065	1.17367
12	2	2	92	22.20	8464	4.52179	22.6104	1.29096
13	2	3	93	24.10	8649	4.53260	22.6745	1.28092

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	31	0.05	19.3671	24.9469	8.3230	1.57771	1.52720
2	31	0.05	13.9384	26.5090	6.1763	1.37410	1.13331
3	31	0.05	21.9401	29.1452	7.6973	1.49300	1.41240
4	31	0.05	22.2980	27.0062	5.9479	1.11672	1.09139
5	31	0.05	22.1861	28.3096	9.4322	1.80038	1.73073
6	31	0.05	20.6562	25.5807	6.1615	1.15940	1.13059
7	31	0.05	22.2780	26.9060	10.2080	1.91504	1.87308
8	31	0.05	20.9663	31.5197	3.5170	0.73324	0.64535
9	31	0.05	18.2525	24.7312	-2.4118	-0.46263	-0.44255
10	31	0.05	19.0789	24.8407	7.2002	1.36793	1.32118
11	31	0.05	20.9128	25.7002	1.8935	0.35579	0.34744
12	31	0.05	19.9775	25.2434	-0.4104	-0.07752	-0.07531
13	31	0.05	20.0620	25.2869	1.4255	0.26911	0.26157

conditional residuals (only white noise) and predicted values using fixed+random effects

```
                    Plot of Resid*Pred.  Symbol used is '*'.

      ,
      ,
   10 ˆ                                         *
      ,                                            *
      ,                          *
      ,                                              *
      ,                         *
      ,              *                 *
      ,                                         *
    5 ˆ
      ,                                *
      ,                                                  *
R     ,                                  *
e     ,                               * *
s     ,                             *   *    *
i     ,
d   0 ˆ                                           *
u     ,                             *
a     ,                                  *      *
l     ,                      *            *
      ,
      ,            *
      ,                                              *
   -5 ˆ                             *     *  *
      ,                               *
      ,                             *       *       *
      ,
      ,                       *                          *
      ,                                         *
      ,
  -10 ˆ
      ,
      Š-ˆ-----------ˆ-----------ˆ-----------ˆ-----------ˆ-----------ˆ-
       18          20          22          24          26          28

                                   Predicted
```


conditional residuals (only white noise) and predicted values using fixed+random effects

The UNIVARIATE Procedure
Variable: Resid (Residual)

(some outputs deleted)

Tests for Normality

Test	--Statistic---		-----p Value------	
Shapiro-Wilk	W	0.94912	Pr < W	0.1156
Kolmogorov-Smirnov	D	0.110991	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.071079	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.498054	Pr > A-Sq	0.2060

```
                      Normal Probability Plot
  11+                                                   ++*
    |                                              *++*+
    |                                         ****++
    |                                       **++
    |                                    +**+
   1+                                +****
    |                             ****
    |                         +++**
    |                      +*****
    |        *   *+***
 -9+     *   ++++
    +----+----+----+----+----+----+----+----+----+----+
         -2        -1         0        +1        +2
```

```
   MIXED MODEL 1: only one error term, as in OLS           11
     marginal residuals (all random components)
   and predicted values using fixed effects only
```

(Not shown as these are the same as cond when no G and R is iid normal)

MIXED MODEL 2:split error into plot (level-2) and tree (level-1)   17

```
                      The Mixed Procedure

                       Model Information

        Data Set                     WORK.TREES2
        Dependent Variable           height
        Covariance Structure         Variance Components
        Subject Effect               plot
        Estimation Method            ML
        Residual Variance Method     Profile
        Fixed Effects SE Method      Model-Based
        Degrees of Freedom Method    Containment


                    Class Level Information

          Class    Levels    Values
          plot          3    1 2 3


                          Dimensions
               Covariance Parameters          2
               Columns in X                   1
               Columns in Z Per Subject       1
               Subjects                       3
               Max Obs Per Subject           13
```

Number of Observations

Number of Observations Read	34
Number of Observations Used	34
Number of Observations Not Used	0

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	214.29118477	
1	2	186.67611539	0.00033195
2	1	186.65306564	0.00001429
3	1	186.65215141	0.00000003
4	1	186.65214940	0.00000000

Convergence criteria met.

MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 18

The Mixed Procedure

Estimated G Matrix

Row	Effect	plot	Col1
1	Intercept	1	22.9657

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
Intercept	plot	22.9657
Residual		10.6647

Fit Statistics

-2 Log Likelihood	186.7
AIC (smaller is better)	192.7
AICC (smaller is better)	193.5
BIC (smaller is better)	189.9

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	23.7300	2.8236	2	8.40	0.0139	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	11.5812	35.8788

The Mixed Procedure

Covariance Matrix
for Fixed Effects

Row	Effect	Col1
1	Intercept	7.9725

Solution for Random Effects

Effect	plot	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	1	5.7510	2.8809	31	2.00	0.0548
Intercept	2	0.000759	2.8678	31	0.00	0.9998
Intercept	3	-5.7517	2.8757	31	-2.00	0.0543

conditional residuals (only white noise) and predicted values
using fixed+random effects

(only first 13 of 34 observations shown)

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	29.4809	1.01727
2	1	2	57	26.40	3249	4.04305	29.4809	1.01727
3	1	3	140	33.24	19600	4.94164	29.4809	1.01727
4	1	4	125	30.60	15625	4.82831	29.4809	1.01727
5	1	5	135	34.68	18225	4.90527	29.4809	1.01727
6	1	6	100	29.28	10000	4.60517	29.4809	1.01727
7	1	7	124	34.80	15376	4.82028	29.4809	1.01727
8	1	8	152	29.76	23104	5.02388	29.4809	1.01727
9	1	9	75	19.08	5625	4.31749	29.4809	1.01727
10	1	10	82	29.16	6724	4.40672	29.4809	1.01727
11	2	1	103	25.20	10609	4.63473	23.7307	0.89529
12	2	2	92	22.20	8464	4.52179	23.7307	0.89529
13	2	3	93	24.10	8649	4.53260	23.7307	0.89529

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	31	0.05	27.4062	31.5557	0.9991	0.32195	0.30593
2	31	0.05	27.4062	31.5557	-3.0809	-0.99283	-0.94343
3	31	0.05	27.4062	31.5557	3.7591	1.21135	1.15108
4	31	0.05	27.4062	31.5557	1.1191	0.36062	0.34267
5	31	0.05	27.4062	31.5557	5.1991	1.67539	1.59203
6	31	0.05	27.4062	31.5557	-0.2009	-0.06475	-0.06153
7	31	0.05	27.4062	31.5557	5.3191	1.71406	1.62877
8	31	0.05	27.4062	31.5557	0.2791	0.08993	0.08545
9	31	0.05	27.4062	31.5557	-10.4009	-3.35168	-3.18492
10	31	0.05	27.4062	31.5557	-0.3209	-0.10342	-0.09828
11	31	0.05	21.9048	25.5567	1.4693	0.46783	0.44991
12	31	0.05	21.9048	25.5567	-1.5307	-0.48741	-0.46874
13	31	0.05	21.9048	25.5567	0.3693	0.11758	0.11307

```
   MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 22
    conditional residuals (only white noise) and predicted values
                      using fixed+random effects

                Plot of Resid*Pred.  Symbol used is '*'.

         ,
        ,
    7.5 ^
        ,
        ,
        ,                                                        *
    5.0 ^                                                        *
        ,
        ,                             *                           *
        ,*
    2.5 ^                             *
        ,
        ,*                            *                           *
        ,*                            *
R   0.0 ^                                                        *
e       ,*                            *                           *
s       ,*                            *
i       ,*                            *
d  -2.5 ^
u       ,                                                        *
a       ,*
l       ,
   -5.0 ^
        ,
        ,
        ,
   -7.5 ^
        ,                             *
        ,
        ,
  -10.0 ^
        ,                                                        *
        ,
        ,
  -12.5 ^
        ,
        Š^---------^---------^---------^---------^---------^---------^
        18        20        22        24        26        28        30

                                   Predicted

NOTE: 11 obs hidden.
```

```
 MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 23
conditional residuals (only white noise) and predicted values
                   using fixed+random effects

                    The UNIVARIATE Procedure
                   Variable:  Resid  (Residual)
```

(some outputs removed)

```
                     Tests for Normality

Test                  --Statistic---    -----p Value------
Shapiro-Wilk          W     0.903479    Pr < W      0.0057
Kolmogorov-Smirnov    D     0.143323    Pr > D      0.0762
Cramer-von Mises      W-Sq  0.149226    Pr > W-Sq   0.0233
Anderson-Darling      A-Sq  0.948388    Pr > A-Sq   0.0159
```

```
 MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 24
  conditional residuals (only white noise) and predicted values
                   using fixed+random effects


 MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 25
  conditional residuals (only white noise) and predicted values
                   using fixed+random effects
```

```
                    Normal Probability Plot
  5.5+                                         +*+  *
     |                                      +++
     |                                   +** *
     |                                ++**
     |                            ******
     |                        ****+
     |                   *****+
     |                ***+++
 -2.5+              **+++
     |             *++
     |          +*+
     |       +++
     |   ++++
     |+++     *
     |
     |
-10.5+    *
      +----+----+----+----+----+----+----+----+----+----+
          -2        -1         0        +1        +2
```

MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 26
marginal residuals (all random components)
and predicted values using fixed effects only

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	23.7300	2.82355
2	1	2	57	26.40	3249	4.04305	23.7300	2.82355
3	1	3	140	33.24	19600	4.94164	23.7300	2.82355
4	1	4	125	30.60	15625	4.82831	23.7300	2.82355
5	1	5	135	34.68	18225	4.90527	23.7300	2.82355
6	1	6	100	29.28	10000	4.60517	23.7300	2.82355
7	1	7	124	34.80	15376	4.82028	23.7300	2.82355
8	1	8	152	29.76	23104	5.02388	23.7300	2.82355
9	1	9	75	19.08	5625	4.31749	23.7300	2.82355
10	1	10	82	29.16	6724	4.40672	23.7300	2.82355
11	2	1	103	25.20	10609	4.63473	23.7300	2.82355
12	2	2	92	22.20	8464	4.52179	23.7300	2.82355
13	2	3	93	24.10	8649	4.53260	23.7300	2.82355

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	31	0.05	17.9713	29.4887	6.7500	1.33258	1.16396
2	31	0.05	17.9713	29.4887	2.6700	0.52711	0.46041
3	31	0.05	17.9713	29.4887	9.5100	1.87746	1.63989
4	31	0.05	17.9713	29.4887	6.8700	1.35627	1.18466
5	31	0.05	17.9713	29.4887	10.9500	2.16174	1.88820
6	31	0.05	17.9713	29.4887	5.5500	1.09568	0.95704
7	31	0.05	17.9713	29.4887	11.0700	2.18543	1.90890
8	31	0.05	17.9713	29.4887	6.0300	1.19044	1.03981
9	31	0.05	17.9713	29.4887	-4.6500	-0.91800	-0.80184
10	31	0.05	17.9713	29.4887	5.4300	1.07199	0.93634
11	31	0.05	17.9713	29.4887	1.4700	0.29021	0.25349
12	31	0.05	17.9713	29.4887	-1.5300	-0.30205	-0.26383
13	31	0.05	17.9713	29.4887	0.3700	0.07305	0.06381

```
   MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 28
              marginal residuals (all random components)
            and predicted values using fixed effects only

               Plot of Resid*Pred.  Symbol used is '*'.

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R   5 ˆ
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i     ,                               *
d     ,                               *
u     ,                               *
a     ,                               *
l   0 ˆ
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  -10 ˆ                               *
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     0.000                         23.730                        47.460

                              Predicted Mean

NOTE: 14 obs hidden.
```

```
   MIXED MODEL 2:split error into plot (level-2) and tree (level-1) 29
           marginal residuals (all random components)
         and predicted values using fixed effects only

                    The UNIVARIATE Procedure
                  Variable:  Resid  (Residual)
```
(some outputs deleted)

```
                       Tests for Normality

   Test                  --Statistic---    -----p Value------
   Shapiro-Wilk          W     0.961976    Pr < W      0.2774
   Kolmogorov-Smirnov    D     0.122965    Pr > D     >0.1500
   Cramer-von Mises      W-Sq  0.053181    Pr > W-Sq  >0.2500
   Anderson-Darling      A-Sq  0.376242    Pr > A-Sq  >0.2500


                      Normal Probability Plot
      11+                                             *+++*
        |                                          *+++
        |                                      ***++
        |                                    **+
        |                                ****
       1+                            ****
        |                         ++**
        |                      +++*
        |                  ++*****
        |           *   *+****
      -9+      *    ++++
         +----+----+----+----+----+----+----+----+----+----+
             -2        -1         0        +1        +2
```

MIXED MODEL 3: fit linear model split error into plot (level-2) and 32

```
                        The Mixed Procedure

                         Model Information

          Data Set                      WORK.TREES2
          Dependent Variable            height
          Covariance Structure          Variance Components
          Subject Effect                plot
          Estimation Method             ML
          Residual Variance Method      Profile
          Fixed Effects SE Method       Model-Based
          Degrees of Freedom Method     Containment


                      Class Level Information

            Class    Levels    Values

            plot          3    1 2 3
```

Dimensions

Covariance Parameters	2
Columns in X	3
Columns in Z Per Subject	1
Subjects	3
Max Obs Per Subject	13

Number of Observations

Number of Observations Read	34
Number of Observations Used	34
Number of Observations Not Used	0

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	211.78758028	
1	2	165.18807686	0.00347301
2	1	164.96916986	0.00077431
3	1	164.92357792	0.00005802
4	1	164.92044850	0.00000042
5	1	164.92042684	0.00000000

The Mixed Procedure

Convergence criteria met.

Estimated G Matrix

Row	Effect	plot	Col1
1	Intercept	1	27.5124

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
Intercept	plot	27.5124
Residual		5.2108

Fit Statistics

-2 Log Likelihood	164.9
AIC (smaller is better)	174.9
AICC (smaller is better)	177.1
BIC (smaller is better)	170.4

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	-63.9111	34.7256	2	-1.84	0.2071	0.05
age	-0.1359	0.09953	29	-1.37	0.1825	0.05
lnage	21.9888	9.7004	29	2.27	0.0310	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-213.32	85.5011
age	-0.3395	0.06763
lnage	2.1492	41.8284

MIXED MODEL 3: fit linear model split error into plot (level-2) and 34

The Mixed Procedure

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3
1	Intercept	1205.87	3.3595	-335.09
2	age	3.3595	0.009907	-0.9518
3	lnage	-335.09	-0.9518	94.0983

Solution for Random Effects

Effect	plot	Estimate	Std Err Pred	DF	t Value	Pr> |t|
Intercept	1	6.2701	3.0845	29	2.03	0.0513
Intercept	2	0.1964	3.0778	29	0.06	0.9496
Intercept	3	-6.4665	3.0804	29	-2.10	0.0446

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
age	1	29	1.87	0.1825
lnage	1	29	5.14	0.0310

MIXED MODEL 3: fit linear model split error into plot (level-2) and 35
conditional residuals (only white noise) and predicted values
using fixed+random effects

(first 13 of 34 shown)

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	28.4925	0.88385
2	1	2	57	26.40	3249	4.04305	23.5121	1.35086
3	1	3	140	33.24	19600	4.94164	31.9880	0.92161
4	1	4	125	30.60	15625	4.82831	31.5352	0.79299
5	1	5	135	34.68	18225	4.90527	31.8680	0.85390
6	1	6	100	29.28	10000	4.60517	30.0270	0.86077
7	1	7	124	34.80	15376	4.82028	31.4945	0.79162
8	1	8	152	29.76	23104	5.02388	32.1650	1.18501
9	1	9	75	19.08	5625	4.31749	27.0998	0.90752
10	1	10	82	29.16	6724	4.40672	28.1103	0.88648
11	2	1	103	25.20	10609	4.63473	24.1955	0.66254
12	2	2	92	22.20	8464	4.52179	23.2074	0.68283
13	2	3	93	24.10	8649	4.53260	23.3092	0.68044

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	29	0.05	26.6849	30.3002	1.98747	0.94431	0.87065
2	29	0.05	20.7493	26.2750	2.88785	1.56939	1.26509
3	29	0.05	30.1031	33.8729	1.25199	0.59949	0.54846
4	29	0.05	29.9133	33.1570	-0.93516	-0.43688	-0.40967
5	29	0.05	30.1216	33.6145	2.81196	1.32828	1.23184
6	29	0.05	28.2665	31.7875	-0.74702	-0.35333	-0.32725
7	29	0.05	29.8754	33.1135	3.30552	1.54386	1.44806
8	29	0.05	29.7414	34.5887	-2.40504	-1.23269	-1.05358
9	29	0.05	25.2437	28.9558	-8.01975	-3.82882	-3.51323
10	29	0.05	26.2972	29.9233	1.04975	0.49903	0.45986
11	29	0.05	22.8404	25.5506	1.00450	0.45984	0.44004
12	29	0.05	21.8109	24.6040	-1.00743	-0.46250	-0.44133
13	29	0.05	21.9176	24.7009	0.79080	0.36292	0.34643

```
MIXED MODEL 3: fit linear model split error into plot (level-2) and 37
    conditional residuals (only white noise) and predicted values
                        using fixed+random effects

                 Plot of Resid*Pred.  Symbol used is '*'.

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e    ,                     *  *        *     *             *
s    ,
i    ,
d -2 ˆ
u    ,                      *               *               *
a    ,                                   *
l    ,                                    *
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  -6 ˆ
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      10          15          20          25          30          35

                                 Predicted
```

```
MIXED MODEL 3: fit linear model split error into plot (level-2) and 38
    conditional residuals (only white noise) and predicted values
                     using fixed+random effects

                      The UNIVARIATE Procedure
                    Variable:  Resid  (Residual)
```
(some outputs removed)

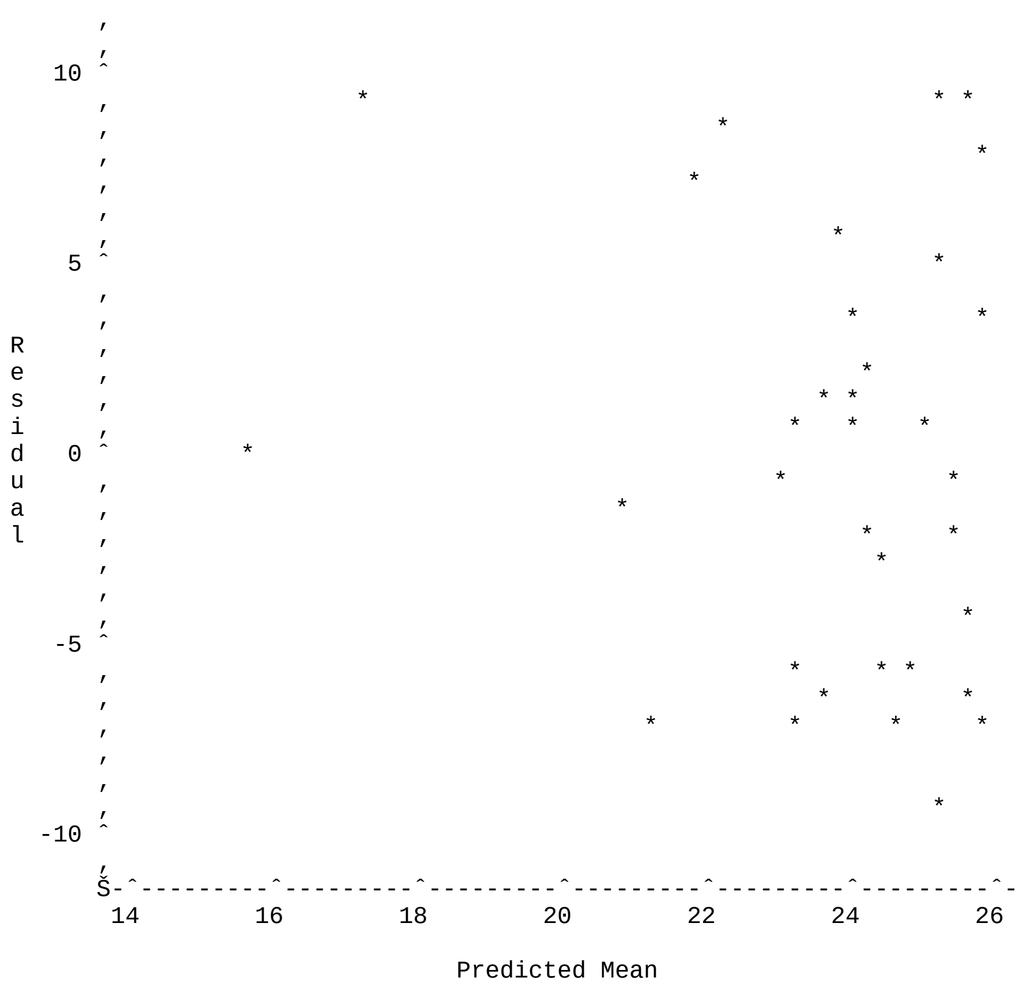
```
                        Tests for Normality

      Test                  --Statistic---    -----p Value------
      Shapiro-Wilk          W     0.900234    Pr < W      0.0047
      Kolmogorov-Smirnov    D     0.147783    Pr > D      0.0587
      Cramer-von Mises      W-Sq  0.100752    Pr > W-Sq   0.1068
      Anderson-Darling      A-Sq  0.702791    Pr > A-Sq   0.0634


                         Normal Probability Plot
    3.5+                                      +++*+  *
       |                                  +**+*
       |                             ******
       |                        *****
       |                 *******
       |               **+++
   -2.5+        *  *+***
       |       +++++
       |   ++++
       |+++
       |
       |
   -8.5+    *
        +----+----+----+----+----+----+----+----+----+----+
            -2        -1         0        +1        +2

MIXED MODEL 3: fit linear model split error into plot (level-2) and 41
             marginal residuals (all random components)
           and predicted values using fixed effects only
```
(only first 13 of 34 shown)

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	22.2225	3.08510
2	1	2	57	26.40	3249	4.04305	17.2421	3.30042
3	1	3	140	33.24	19600	4.94164	25.7180	3.11928
4	1	4	125	30.60	15625	4.82831	25.2651	3.06706
5	1	5	135	34.68	18225	4.90527	25.5980	3.09380
6	1	6	100	29.28	10000	4.60517	23.7570	3.07300
7	1	7	124	34.80	15376	4.82028	25.2244	3.06581
8	1	8	152	29.76	23104	5.02388	25.8950	3.22339
9	1	9	75	19.08	5625	4.31749	20.8297	3.10319
10	1	10	82	29.16	6724	4.40672	21.8402	3.08850
11	2	1	103	25.20	10609	4.63473	23.9991	3.07046
12	2	2	92	22.20	8464	4.52179	23.0110	3.07935
13	2	3	93	24.10	8649	4.53260	23.1128	3.07859

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	29	0.05	15.9127	28.5322	8.25752	1.71418	1.44352
2	29	0.05	10.4920	23.9922	9.15791	1.96004	1.60091
3	29	0.05	19.3383	32.0976	7.52204	1.56868	1.31495
4	29	0.05	18.9923	31.5380	5.33489	1.10483	0.93261
5	29	0.05	19.2704	31.9255	9.08202	1.88752	1.58765
6	29	0.05	17.4720	30.0420	5.52303	1.14469	0.96549
7	29	0.05	18.9541	31.4947	9.57557	1.98273	1.67393
8	29	0.05	19.3024	32.4876	3.86501	0.81786	0.67565
9	29	0.05	14.4830	27.1764	-1.74970	-0.36410	-0.30587
10	29	0.05	15.5235	28.1569	7.31980	1.52020	1.27959
11	29	0.05	17.7193	30.2789	1.20089	0.24881	0.20993
12	29	0.05	16.7131	29.3090	-0.81103	-0.16823	-0.14178
13	29	0.05	16.8164	29.4092	0.98719	0.20475	0.17257

MIXED MODEL 3: fit linear model split error into plot (level-2) and 43
marginal residuals (all random components)
and predicted values using fixed effects only

Plot of Resid*Pred. Symbol used is '*'.



```
MIXED MODEL 3: fit linear model split error into plot (level-2) and 44
             marginal residuals (all random components)
           and predicted values using fixed effects only

                      The UNIVARIATE Procedure
                    Variable:  Resid  (Residual)
```
(some outputs removed)

```
                        Tests for Normality

      Test                  --Statistic---    -----p Value------
      Shapiro-Wilk          W     0.941072    Pr < W      0.0664
      Kolmogorov-Smirnov    D     0.120962    Pr > D     >0.1500
      Cramer-von Mises      W-Sq  0.068097    Pr > W-Sq  >0.2500
      Anderson-Darling      A-Sq  0.553837    Pr > A-Sq   0.1456


                         Normal Probability Plot
            9+                                      * *++*+  *
             |                                    **+++
             |                                  **+
            3+                              +***
             |                          *****
             |                       +***
           -3+                    ++**
             |                ++****
             |         *   *+****
           -9+     *   ++++
              +----+----+----+----+----+----+----+----+----+----+
                  -2        -1         0        +1        +2
```

MODEL 4: linear model, split error into plot (level-2) and tree (le 47
and allow for compound symmetry error matrix-- same as varying interce

```
                         The Mixed Procedure

                          Model Information

         Data Set                     WORK.TREES2
         Dependent Variable           height
         Covariance Structure         Compound Symmetry
         Subject Effect               plot
         Estimation Method            ML
         Residual Variance Method     Profile
         Fixed Effects SE Method      Model-Based
         Degrees of Freedom Method    Between-Within


                       Class Level Information

           Class    Levels    Values
           plot          3    1 2 3
```

Dimensions

Covariance Parameters	2
Columns in X	3
Columns in Z	0
Subjects	3
Max Obs Per Subject	13

Number of Observations

Number of Observations Read	34
Number of Observations Used	34
Number of Observations Not Used	0

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	211.78758028	
1	2	165.18807686	0.00347301
2	1	164.96916986	0.00077431
3	1	164.92357792	0.00005802
4	1	164.92044850	0.00000042
5	1	164.92042684	0.00000000

Convergence criteria met.

MODEL 4: linear model, split error into plot (level-2) and tree (le 48
and allow for compound symmetry error matrix-- same as varying interce

The Mixed Procedure

Estimated R Matrix for plot 1

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	32.7232	27.5124	27.5124	27.5124	27.5124	27.5124
2	27.5124	32.7232	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	32.7232	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	32.7232	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124	32.7232	27.5124
6	27.5124	27.5124	27.5124	27.5124	27.5124	32.7232
7	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
8	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
9	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
10	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124

Estimated R Matrix for plot 1

Row	Col7	Col8	Col9	Col10
1	27.5124	27.5124	27.5124	27.5124
2	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124
6	27.5124	27.5124	27.5124	27.5124
7	32.7232	27.5124	27.5124	27.5124
8	27.5124	32.7232	27.5124	27.5124
9	27.5124	27.5124	32.7232	27.5124
10	27.5124	27.5124	27.5124	32.7232

MODEL 4: linear model, split error into plot (level-2) and tree (le 49
and allow for compound symmetry error matrix-- same as varying interce

The Mixed Procedure

Estimated R Matrix for plot 2

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	32.7232	27.5124	27.5124	27.5124	27.5124	27.5124
2	27.5124	32.7232	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	32.7232	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	32.7232	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124	32.7232	27.5124
6	27.5124	27.5124	27.5124	27.5124	27.5124	32.7232
7	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
8	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
9	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
10	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
11	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
12	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
13	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124

Estimated R Matrix for plot 2

Row	Col7	Col8	Col9	Col10	Col11	Col12
1	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
2	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
6	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
7	32.7232	27.5124	27.5124	27.5124	27.5124	27.5124
8	27.5124	32.7232	27.5124	27.5124	27.5124	27.5124
9	27.5124	27.5124	32.7232	27.5124	27.5124	27.5124
10	27.5124	27.5124	27.5124	32.7232	27.5124	27.5124
11	27.5124	27.5124	27.5124	27.5124	32.7232	27.5124
12	27.5124	27.5124	27.5124	27.5124	27.5124	32.7232
13	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124

Row	Col13
1	27.5124
2	27.5124
3	27.5124
4	27.5124
5	27.5124
6	27.5124
7	27.5124
8	27.5124
9	27.5124
10	27.5124
11	27.5124
12	27.5124
13	32.7232

Estimated R Matrix for plot 3

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	32.7232	27.5124	27.5124	27.5124	27.5124	27.5124
2	27.5124	32.7232	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	32.7232	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	32.7232	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124	32.7232	27.5124
6	27.5124	27.5124	27.5124	27.5124	27.5124	32.7232
7	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124

Estimated R Matrix for plot 3

Row	Col7	Col8	Col9	Col10	Col11
1	27.5124	27.5124	27.5124	27.5124	27.5124
2	27.5124	27.5124	27.5124	27.5124	27.5124
3	27.5124	27.5124	27.5124	27.5124	27.5124
4	27.5124	27.5124	27.5124	27.5124	27.5124
5	27.5124	27.5124	27.5124	27.5124	27.5124
6	27.5124	27.5124	27.5124	27.5124	27.5124
7	32.7232	27.5124	27.5124	27.5124	27.5124

Estimated R Matrix for plot 3

Row	Col1	Col2	Col3	Col4	Col5	Col6
8	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
9	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
10	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124
11	27.5124	27.5124	27.5124	27.5124	27.5124	27.5124

Estimated R Matrix for plot 3

Row	Col7	Col8	Col9	Col10	Col11
8	27.5124	32.7232	27.5124	27.5124	27.5124
9	27.5124	27.5124	32.7232	27.5124	27.5124
10	27.5124	27.5124	27.5124	32.7232	27.5124
11	27.5124	27.5124	27.5124	27.5124	32.7232

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
CS	plot	27.5124
Residual		5.2108

Fit Statistics

-2 Log Likelihood	164.9
AIC (smaller is better)	174.9
AICC (smaller is better)	177.1
BIC (smaller is better)	170.4

(This tests the covariance matrix against an iid error covariance matrix)

Null Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
1	46.87	<.0001

MODEL 4: linear model, split error into plot (level-2) and tree (le 52
and allow for compound symmetry error matrix-- same as varying interce

The Mixed Procedure

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr> |t|	Alpha
Intercept	-63.9111	34.7256	2	-1.84	0.2071	0.05
age	-0.1359	0.09953	29	-1.37	0.1825	0.05
lnage	21.9888	9.7004	29	2.27	0.0310	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-213.32	85.5011
age	-0.3395	0.06763
lnage	2.1492	41.8284

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3
1	Intercept	1205.87	3.3595	-335.09
2	age	3.3595	0.009907	-0.9518
3	lnage	-335.09	-0.9518	94.0983

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
age	1	29	1.87	0.1825
lnage	1	29	5.14	0.0310

MODEL 4: linear model, split error into plot (level-2) and tree (le 53
and allow for compound symmetry error matrix-- same as varying interce
conditional residuals (only white noise) and predicted values
using fixed+random effects

(only first 13 of 34 observations shown)

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	22.2225	3.08510
2	1	2	57	26.40	3249	4.04305	17.2421	3.30042
3	1	3	140	33.24	19600	4.94164	25.7180	3.11928
4	1	4	125	30.60	15625	4.82831	25.2651	3.06706
5	1	5	135	34.68	18225	4.90527	25.5980	3.09380
6	1	6	100	29.28	10000	4.60517	23.7570	3.07300
7	1	7	124	34.80	15376	4.82028	25.2244	3.06581
8	1	8	152	29.76	23104	5.02388	25.8950	3.22339
9	1	9	75	19.08	5625	4.31749	20.8297	3.10319
10	1	10	82	29.16	6724	4.40672	21.8402	3.08850
11	2	1	103	25.20	10609	4.63473	23.9991	3.07046
12	2	2	92	22.20	8464	4.52179	23.0110	3.07935
13	2	3	93	24.10	8649	4.53260	23.1128	3.07859

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	31	0.05	15.9304	28.5146	8.25752	1.71418	1.44352
2	31	0.05	10.5108	23.9734	9.15791	1.96004	1.60091
3	31	0.05	19.3561	32.0798	7.52204	1.56868	1.31495
4	31	0.05	19.0098	31.5204	5.33489	1.10483	0.93261
5	31	0.05	19.2881	31.9078	9.08202	1.88752	1.58765
6	31	0.05	17.4895	30.0244	5.52303	1.14469	0.96549
7	31	0.05	18.9717	31.4772	9.57557	1.98273	1.67393
8	31	0.05	19.3208	32.4691	3.86501	0.81786	0.67565
9	31	0.05	14.5007	27.1587	-1.74970	-0.36410	-0.30587
10	31	0.05	15.5412	28.1392	7.31980	1.52020	1.27959
11	31	0.05	17.7369	30.2613	1.20089	0.24881	0.20993
12	31	0.05	16.7307	29.2914	-0.81103	-0.16823	-0.14178
13	31	0.05	16.8340	29.3916	0.98719	0.20475	0.17257

```
MODEL 4: linear model, split error into plot (level-2) and tree (le 55
and allow for compound symmetry error matrix-- same as varying interce
    conditional residuals (only white noise) and predicted values
                      using fixed+random effects

               Plot of Resid*Pred.  Symbol used is '*'.

       ,
       ,
    10 ^
       ,                 *                                       * *
       ,                                          *
       ,                                                            *
       ,                                        *
       ,
       ,                                                  *
     5 ^                                                         *
       ,
       ,                                                   *        *
R      ,
e      ,                                                    *
s      ,                                                 *  *
i      ,                                               *   *    *
d    0 ^         *
u      ,                                              *           *
a      ,                                   *
l      ,                                                    *     *
       ,                                                     *
       ,
       ,                                                           *
    -5 ^
       ,                                               *     * *
       ,                                                 *         *
       ,                                     *         *      *     *
       ,
       ,
       ,                                                         *
   -10 ^
       ,
       Š-^---------^---------^---------^---------^---------^---------^-
        14        16        18        20        22        24        26

                                   Predicted
```

```
MODEL 4: linear model, split error into plot (level-2) and tree (le 56
and allow for compound symmetry error matrix-- same as varying interce
    conditional residuals (only white noise) and predicted values
                      using fixed+random effects

                       The UNIVARIATE Procedure
                     Variable:  Resid  (Residual)
```

(some outputs removed)

Tests for Normality

Test	--Statistic---		-----p Value------	
Shapiro-Wilk	W	0.941072	Pr < W	0.0664
Kolmogorov-Smirnov	D	0.120962	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.068097	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.553837	Pr > A-Sq	0.1456

```
                 Normal Probability Plot
 9+                                     *  *++*+   *
  |                                   **+++
  |                                 **+
 3+                               +***
  |                           *****
  |                        +***
-3+                     ++**
  |                 ++****
  |           *  *+****
-9+       *   ++++
   +----+----+----+----+----+----+----+----+----+----+
       -2        -1         0        +1        +2
```

MODEL 4: linear model, split error into plot (level-2) and tree (le 59
and allow for compound symmetry error matrix-- same as varying interce
marginal residuals (all random components)
and predicted values using fixed effects only

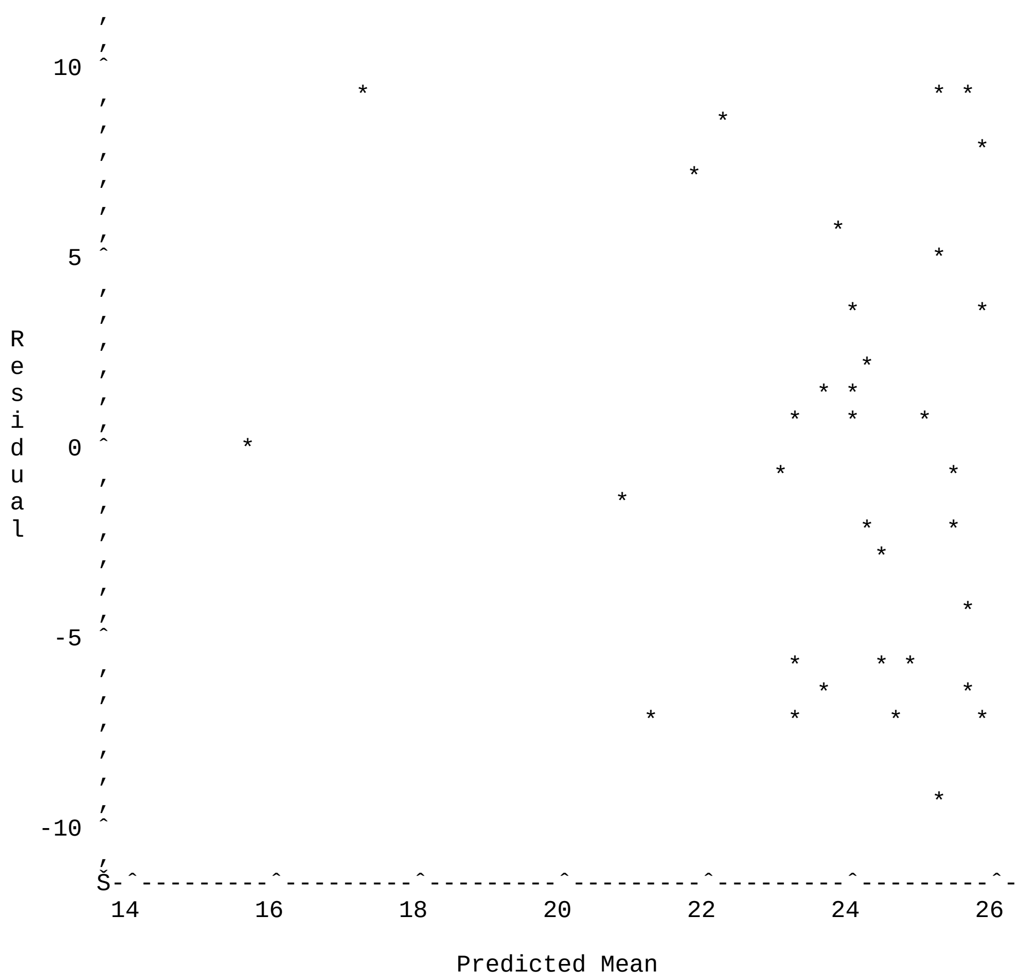
(only 13 of 34 observations shown)

Obs	plot	Tree	age	height	agesq	lnage	Pred	StdErr Pred
1	1	1	85	30.48	7225	4.44265	22.2225	3.08510
2	1	2	57	26.40	3249	4.04305	17.2421	3.30042
3	1	3	140	33.24	19600	4.94164	25.7180	3.11928
4	1	4	125	30.60	15625	4.82831	25.2651	3.06706
5	1	5	135	34.68	18225	4.90527	25.5980	3.09380
6	1	6	100	29.28	10000	4.60517	23.7570	3.07300
7	1	7	124	34.80	15376	4.82028	25.2244	3.06581
8	1	8	152	29.76	23104	5.02388	25.8950	3.22339
9	1	9	75	19.08	5625	4.31749	20.8297	3.10319
10	1	10	82	29.16	6724	4.40672	21.8402	3.08850
11	2	1	103	25.20	10609	4.63473	23.9991	3.07046
12	2	2	92	22.20	8464	4.52179	23.0110	3.07935
13	2	3	93	24.10	8649	4.53260	23.1128	3.07859

Obs	DF	Alpha	Lower	Upper	Resid	Student Resid	Pearson Resid
1	31	0.05	15.9304	28.5146	8.25752	1.71418	1.44352
2	31	0.05	10.5108	23.9734	9.15791	1.96004	1.60091
3	31	0.05	19.3561	32.0798	7.52204	1.56868	1.31495
4	31	0.05	19.0098	31.5204	5.33489	1.10483	0.93261
5	31	0.05	19.2881	31.9078	9.08202	1.88752	1.58765
6	31	0.05	17.4895	30.0244	5.52303	1.14469	0.96549
7	31	0.05	18.9717	31.4772	9.57557	1.98273	1.67393
8	31	0.05	19.3208	32.4691	3.86501	0.81786	0.67565
9	31	0.05	14.5007	27.1587	-1.74970	-0.36410	-0.30587
10	31	0.05	15.5412	28.1392	7.31980	1.52020	1.27959
11	31	0.05	17.7369	30.2613	1.20089	0.24881	0.20993
12	31	0.05	16.7307	29.2914	-0.81103	-0.16823	-0.14178
13	31	0.05	16.8340	29.3916	0.98719	0.20475	0.17257

MODEL 4: linear model, split error into plot (level-2) and tree (le 61
and allow for compound symmetry error matrix-- same as varying interce
marginal residuals (all random components)
and predicted values using fixed effects only

Plot of Resid*Pred. Symbol used is '*'.



MODEL 4: linear model, split error into plot (level-2) and tree (le 62
and allow for compound symmetry error matrix-- same as varying interce
marginal residuals (all random components)
and predicted values using fixed effects only

The UNIVARIATE Procedure
Variable: Resid (Residual)

(some outputs removed)

Tests for Normality

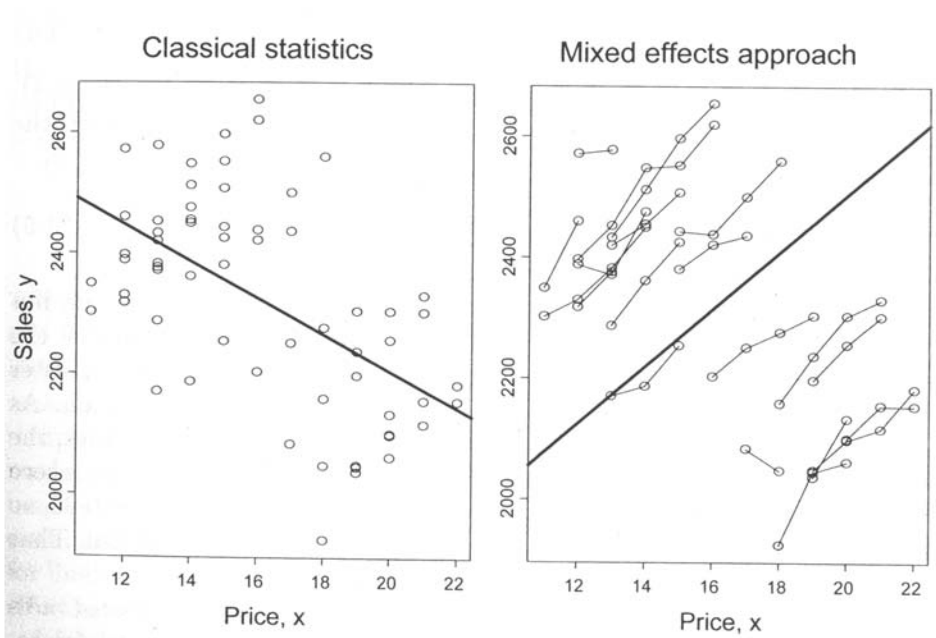
Test	--Statistic---		-----p Value------	
Shapiro-Wilk	W	0.941072	Pr < W	0.0664
Kolmogorov-Smirnov	D	0.120962	Pr > D	>0.1500
Cramer-von Mises	W-Sq	0.068097	Pr > W-Sq	>0.2500
Anderson-Darling	A-Sq	0.553837	Pr > A-Sq	0.1456

MODEL 4: linear model, split error into plot (level-2) and tree (le 63
and allow for compound symmetry error matrix-- same as varying interce
marginal residuals (all random components)
and predicted values using fixed effects only

```
                 Normal Probability Plot
 9+                                     * *++*+  *
  |                                   **+++
  |                                 **+
 3+                             +***
  |                         *****
  |                      +***
-3+                   ++**
  |               ++****
  |        *  *+****
-9+    *   ++++
   +----+----+----+----+----+----+----+----+----+----+
       -2        -1         0        +1        +2
```

Observations in Time, Grouped by Subject

- As noted, hierarchical models can be considered as clustered data, where data at level-1 are grouped by level-2
- A specific kind of hierarchical data is where we have subjects and these are measured over time
    - y is a measure of subject j at time i.
    - This kind of data is very common and has been termed: Panel data; longitudinal data; cross-sectional data.
- Care must be taking into account the nature of the times within subject measures in specifying the $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$ part of the model:



<u>Model 1: One error term with no **Zu**</u>

- The could be considered (1) a GLS model where the error covariance matrix is block diagonal, or (2) as a mixed linear model
- When there is one component, the $\mathbf{Zu}$ matrix is a zero matrix and drops out of the model. This then becomes:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad \text{Var}(\boldsymbol{\varepsilon}) = \mathbf{R}$$

Where the matrices **Y, X**, $\boldsymbol{\beta}$, and $\boldsymbol{\varepsilon}$ are same as for the OLS (and GLS) model, and **R** is represents the error covariance matrix.

- The **R** matrix is block diagonal with:
  - with serial correlation between measures of a subject,
  - possible heteroskedasticity between subjects;
  - subjects are uncorrelated.
- Each block in the **R** block diagonal matrix:
  - represents a submatrix of the error covariance matrix for the measures for the subject.
  - The off-block diagonal values (between subjects) are all equal to 0, indicating no correlation between subjects.
- <u>This will result in consistent estimates of the standard errors of coefficients in your model, whereas OLS estimates of standard errors will be biased and not consistent</u>.

Using 2 subjects (e.g., trees), 2 measures in time per subject, and assuming no correlation between individuals, the error covariance matrix becomes:

$$\sum = \mathbf{R} = \begin{bmatrix} \sigma^2_{\varepsilon_{11}} & \sigma_{\varepsilon_{11}\varepsilon_{12}} & 0 & 0 \\ \sigma_{\varepsilon_{12}\varepsilon_{11}} & \sigma^2_{\varepsilon_{12}} & 0 & 0 \\ 0 & 0 & \sigma^2_{\varepsilon_{21}} & \sigma_{\varepsilon_{21}\varepsilon_{22}} \\ 0 & 0 & \sigma_{\varepsilon_{22}\varepsilon_{21}} & \sigma^2_{\varepsilon_{22}} \end{bmatrix}$$

We then need to specify only the error covariance for each subject. This could be AR(1) for each subject, for example, with correlations the same for all subjects, and the variances the same for all subjects. This then becomes:

$$\sum = \frac{\sigma_v^{\ 2}}{(1-\rho)} \begin{bmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & \rho & 1 \end{bmatrix}$$

Only two parameters are needed to obtain this more simple error covariance matrix. You can also specify that the parameters change for different subjects. Then you would need <u>2 parameters for each subject (hopefully not needed!).</u>

Model 2: Treat this as a Hierarchical or Random Coefficients Model, Where $\mathbf{Zu}$ represents the Subject Random Effects

For each observation, using *t* for time (individual) and *j* for tree (subject) and including only one explanatory variable and only two trees (level-2), the model is:

$$
\begin{aligned}
y_{tj} &= \beta_0 + \beta_1 x_{1tj} + u_1 z_{1tj} + u_2 z_{2tj} + u_1 z_{1tj} x_{1tj} + u_2 z_{2tj} x_{1tj} + \varepsilon_{ij} \\
&= X_{tj}\beta + Z_j u + \varepsilon_{tj}
\end{aligned}
$$

$$
z_{1tj} \begin{cases} 1 \text{ for subject } 1 \\ 0 \text{ for subject } 2 \end{cases}
$$

$$
z_{2tj} \begin{cases} 0 \text{ for subject } 1 \\ 1 \text{ for subject } 2 \end{cases}
$$

$$
\text{and } \varepsilon_{tj} \text{ are correlated in time, but only for each subject.}
$$

- This allows for different intercepts for different “panels” (subjects in SAS), and for different slopes.
- The design matrix, **Z**, in this case is just dummy variables that alter the intercept for different subjects (*random intercept*) and columns that are interactions between dummy variables and the *x* (*random slope*).
- Because these are measured over time, however, there is correlation among time measures, and the R matrix for this is:

$$
\Sigma = \mathbf{R} = \begin{bmatrix}
\sigma^2_{\varepsilon_{11}} & \sigma_{\varepsilon_{11}\varepsilon_{12}} & 0 & 0 \\
\sigma_{\varepsilon_{12}\varepsilon_{11}} & \sigma^2_{\varepsilon_{12}} & 0 & 0 \\
0 & 0 & \sigma^2_{\varepsilon_{21}} & \sigma_{\varepsilon_{21}\varepsilon_{22}} \\
0 & 0 & \sigma_{\varepsilon_{22}\varepsilon_{21}} & \sigma^2_{\varepsilon_{22}}
\end{bmatrix}
$$

- We will get estimates of (i) the intercepts and slopes for the population, (ii) for each subject in the sample, AND (ii) estimates of the variances due to subjects and due to individuals.
- Accounting for the time correlations in the R matrix will then result in consistent estimates of standard errors of these coefficients also, and we get an estimate of the strength of this correlation in time, within subjects.
- It is also possible to get predictions forward in time for individual subjects (see Yang and LeMay, CFJR).

Example: mixed_panel_data.sas
Data:

- Some of the trees in trees_over_time.xls.
- Height (*httot*) and dbh (*dbhob*) measures for each tree at several times (*ageper*).
- 581 observations representing 27 trees.
- Time measures are not the same for each tree.

OLS: One equation for all trees and measures combined
PROC REG ...;
model httot=dbh dbhsq dbhcu/. . .;

X has 4 columns; $\boldsymbol{\beta}$ has 4 values.

Residuals will be tree + measures within tree errors combined and are considered to be normal, iid.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$. (i.e., Pred. height given dbh).
The **R** matrix is iid with only one parameter, MSE estimating the variance of the combined errors.
There is no **G** matrix with OLS.

OLS 2: Fit each tree separately.
PROC REG ...;
model httot=dbhob dbhsq dbhcu/. . .;
by treeid;

- This will give the same coefficients for each tree as including dummy variables and all interactions.
- We can use this
    - to look at the variability in coefficients among trees.
    - To see what the residual plots look like when tree to tree variability in the model is accounted for.
- There will still be correlated error terms for each tree, however.

Mixed Model: Repeat of OLS
PROC MIXED ...;
model htot=dbhob dbhsq dbhcu/solution. . .;
(no random nor repeated statements)

X has 3 columns; $\boldsymbol{\beta}$ has 3 values; No **Z** matrix.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$ (conditional and marginal predicted values and residuals are the same)
Residuals will be due to tree + time, as OLS, and are considered to be normal, iid.
The **R** matrix is considered to be iid with only one parameter, the estimated variance of the combined errors.
There is no **G** matrix.
Therefore, this is only one covariance parameter (CovParms), and this represented the diagonal values on the n X n **R** matrix.

Mixed Model 1: Include correlated errors within trees
PROC MIXED ...;

model htot=dbhob dbhsq dbhcu/solution. . .;
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;

X has 3 columns; $\boldsymbol{\beta}$ has 3 values; No **Z** matrix.

Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$ (conditional and marginal residuals are the same)
Residuals will be plot + tree errors combined (conditional and marginal residuals are the same) and are considered independent between trees, but correlated between time measures within trees.
The **R** matrix: Autocorrelation similar to AR(1), but the power will be a function of the distance between observations. This is really a spatial autocorrelation (correlation^distance). We will restrict the correlation and the remaining error variance to be the same for every tree, to reduce the number of parameters for which to search.
There is no **G** matrix.

Therefore, there will be two covariance parameters (CovParms), the error variance, and the correlation, used to create a block diagonal **R** matrix. This will be printed for two trees.

Mixed Model 2: Allow some coefficients to vary between trees;
PROC MIXED ...;
class treeid;
model httot=dbhob dbhsq dbhcu/ solution cl covb residual outp=cond2 residual outpm=marg2 residual;
random intercept dbhob/ subject=treeid type=un solution g;
parms (3)(0)(0.6)(16);

X has 3 columns; $\boldsymbol{\beta}$ has 3 values;
Z is columns of 1 or 0's, 1 for each tree, and interactions between dummy variables and the *dbhob* (*random slope*).

Marginal Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$. Same as OLS.
Marginal Residuals Same as OLS.
Conditional Predicted values will be the fixed part of the equation plus added change in intercept and slope with *dbhob* for each tree (estimates of tree level heights given dbh), $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$.
Conditional Residuals is time errors only, representing the difference between the measured tree height at a time, and that estimated for that tree.
The **R** matrix is iid with only one parameter, the estimated variance of the tree heights within a tree.
The **G** matrix has three values, the variance of the intercept between trees, variance of the slope for dbh and covariance between the intercept and the slope.

Therefore, there will have 4 covariance parameters (called *CovParms*), representing the error variance on the diagonal of the n X n **R** matrix, and all elements of the 2 X 2 **G** matrix.

Mixed Model 3: Allow some coefficients to vary between trees, and allow for correlation in time
PROC MIXED ...;

```
class treeid;
model httot=dbhob dbhsq dbhcu/ solution cl covb residual outp=cond2 residual outpm=marg2
residual;
random intercept dbhob/ subject=treeid type=un solution g;
parms (6)(0)(1)(0.9)(2);
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
```

X has 3 columns; $\boldsymbol{\beta}$ has 3 values;

Z is columns of 1 or 0's, 1 for each tree, and interactions between dummy variables and the *dbhob* (*random slope*).

Marginal Predicted values will be the fixed part of the equation, $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta}$. Same as OLS.

Marginal Residuals Same as OLS.

Conditional Predicted values will be the fixed part of the equation plus added change in intercept and slope with *dbhob* for each tree (estimates of tree level heights given dbh), $\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$.

Conditional Residuals is time errors only, representing the difference between the measured tree height at a time, and that estimated for that tree.

The **R** matrix: Autocorrelation similar to AR(1), but the power will be a function of the distance between observations. This is really a spatial autocorrelation (correlation^distance). We will restrict the correlation and the remaining error variance to be the same for every tree, to reduce the number of parameters for which to search.

The **G** matrix has three values, the variance of the intercept between trees, variance of the slope for dbh and covariance between the intercept and the slope.

Therefore, there will have 5 covariance parameters (called *CovParms*), representing the error variance and the correlation representing the blocks on the block diagonal of the *n* X *n* **R** matrix, and all elements of the 2 X 2 **G** matrix.

Mixed Models 4 and 5: Allow some coefficients to vary between trees, and allow for correlation in time

M4: Same as M3, but allow for varying slopes with dbhob and dbhsq only [not the intercept]. 5 parameters in CovParms.

```
random dbhob dbhsq/ subject=treeid type=un solution g;
parms (0.1)(0)(0.002)(0.9)(2);
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
```

M5: Same as M4, but allow for varying slopes with dbhob, dbhsq, and dbhcu [not the intercept]. 8 parameters in CovParms. NOTE: This one is hard to find a solution for. Altered the convergence criterion to be less stringent in order to find a solution.

```
random dbhob dbhsq dbhcu/ subject=treeid type=un solution g;
parms (0.2)(0)(0.002)(0)(0)(0)(0.9)(2);
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
```

```
PROC IMPORT OUT= WORK.panel_data
DATAFILE= "E:\FRST530\datasets\trees_over_time.xls"
DBMS=EXCEL REPLACE;
SHEET="panel_data";   GETNAMES=YES;    MIXED=NO;
SCANTEXT=YES;    USEDATE=YES;    SCANTIME=YES;
RUN;
options ls=70 ps=50 pageno=1 nodate;

title1 ' ';

data panel_data2;
set panel_data;
dbhsq=dbhob**2;
dbhcu=dbhob**3;
lndbh=log(dbhob);
lnht=log(httot);
* create a unique tree identifier using plot and tree;
treeid=(plot*100+tree);
*select a few trees for the example;
if treeid ge 1000 then delete;
run;

proc sort data=panel_data2 out=sorted;
by treeid ageper;
run;

proc means data=sorted n noprint;
var httot;
by treeid;
output out=measures n=nperiods;
run;
```

```
/* plot the height and dbh over time by tree */
GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=4 IN MINOR=NONE ORDER=0 TO 70 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM A=90 R=0 F=SWISS
'height') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 110 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM F=SWISS 'dbh') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=black L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C  '  ' ;
TITLE2 C=BLACK F=SWISS H=0.50 CM J=C 'height and diameter
over time' ;
TITLE3 '  ';
* FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM ' ';
* FOOTNOTE2 ;
* FOOTNOTE3 ;
PROC GPLOT DATA = SORTED ;
        PLOT httot * dbhob=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;

PROC reg data=panel_data2;
title1 'OLS fit -- all trees combined';
model httot=dbhob dbhsq dbhcu;
*model lnht=dbhob lndbh;
output out=pout r=resid p=yhat;
run;

proc plot data=pout;
plot yhat*httot='*';
plot resid*yhat='*';
plot resid*ageper='*';
run;

PROC SORT DATA = pout out=sorted2;
        BY treeid ageper;
RUN ;
```

```
/* plot the common prediction line */
GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=4 IN MINOR=NONE ORDER=0 TO 70 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM A=90 R=0 F=SWISS
'predicted height') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 110 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM F=SWISS 'dbh') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK V=star;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'predicted height
versus dbh' ;
TITLE3 '  ';
*FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM ' ' ;
*FOOTNOTE2 ;
*FOOTNOTE3 ;
PROC GPLOT DATA = SORTED2;
        PLOT yhat * dbhob=1/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;

/* look at variations in coefficients based on OLS fit of
each tree, but don't print out the results*/
Proc reg data=panel_data2 outest=coeffs no print;
title1 'OLS by tree -- same as if plot was included as dummy
variables';
title2 'along with all interactions';
model httot=dbhob dbhsq dbhcu;
output out=pout2 p=yhat r=resid;
by treeid;
run;

proc plot data=pout2;
plot yhat*httot='*';
plot resid*yhat='*';
plot resid*ageper='*';
run;

proc corr data=coeffs cov;
var intercept dbhob dbhsq dbhcu;
with intercept dbhob dbhsq dbhcu;
run;
```

```
* fit model using PROC MIXED, assuming not panel data, same
as one OLS fit for all data combined;
proc mixed data=sorted method=reml;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL: as OLS';
*     class treeid;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond residual outpm=marg residual;
run;

* use the marginal means out of MIXED next in plotting
(broad-- population averaged) -- all random components in
residual.  Same as conditional as there is only one error
term;
proc plot data=marg;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
     plot pred*httot='*';
     plot resid*ageper='*';
run;
proc univariate data=marg plot normal;
          var resid;
run;

* fit model using PROC MIXED, allowing for correlated errors
within subjects;
proc mixed data=sorted method=reml maxiter=300;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL 1: correlated error terms';
class treeid ageper;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond1 residual outpm=marg1 residual;
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged) -- all random components in
residual.  Conditional the same as there is only one error
term;
proc plot data=marg1;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg1 plot normal;
          var resid;
run;
```

```
* fit random coefficients model using PROC MIXED, assuming
iid white noise;
proc mixed data=sorted method=reml maxiter=400 nobound;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL 2: varying slopes';
class treeid;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond2 residual outpm=marg2 residual;
random intercept dbhob/ subject=treeid type=un solution g;
parms (3)(0)(0.6)(16);
run;

* use the conditional means out of MIXED first in plotting
(intermediate or narrow)
-- only the last error term as the residual;
proc plot data=cond2;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
       plot resid*pred='*';
       plot pred*httot='*';
       plot resid*ageper='*';
run;
proc univariate data=cond2 plot normal;
          var resid;
run;

* use the marginal means out of MIXED next in plotting
(broad-- population averaged)
-- all random components in residual;
proc plot data=marg2;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg2 plot normal;
          var resid;
run;

* fit random coefficients model using PROC MIXED, assuming
correlated errors within trees;
proc mixed data=sorted method=reml maxiter=500;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL 3: varying slopes+correlated errors';
class treeid ageper;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond3 residual outpm=marg3 residual;
random intercept dbhob/ subject=treeid type=un solution g;
parms (6)(0)(1)(0.9)(2);
*     parms (0)(0)(0.6)(0.9)(4)/ hold=1,2,3,4,5;
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
run;
```

```
* use the conditional means out of MIXED first in plotting
(intermediate or narrow)
-- only the last error term as the residual;
proc plot data=cond3;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
     plot resid*pred='*';
     plot pred*httot='*';
     plot resid*ageper='*';
run;
proc univariate data=cond3 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)
-- all random components in residual;
proc plot data=marg3;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg3 plot normal;
          var resid;
run;

proc sort data=cond3 out=sorted3;
by treeid ageper;
run;
```

```
* plot the tree fits using the predicted conditional means;
GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=4 IN MINOR=NONE ORDER=0 TO 70 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM A=90 R=0 F=SWISS
'predicted height') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 110 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM F=SWISS 'dbh') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=black L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'predicted height
versus dbh' ;
TITLE3 '  ';
*FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM ' ' ;
*FOOTNOTE2 ;
*FOOTNOTE3 ;
PROC GPLOT DATA = SORTED3;
        PLOT pred * dbhob=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;

* change which coefficents are allowed to be random;
* fit random coefficients model using PROC MIXED, assuming
correlated errors within trees;
proc mixed data=sorted method=reml maxiter=500;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL 4: varying slopes+correlated errors';
class treeid ageper;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond4 residual outpm=marg4 residual;
random dbhob dbhsq/ subject=treeid type=un solution g;
parms (0.1)(0)(0.002)(0.9)(2);
*     parms (0)(0)(0.6)(0.9)(4)/ hold=1,2,3,4,5;
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
run;
```

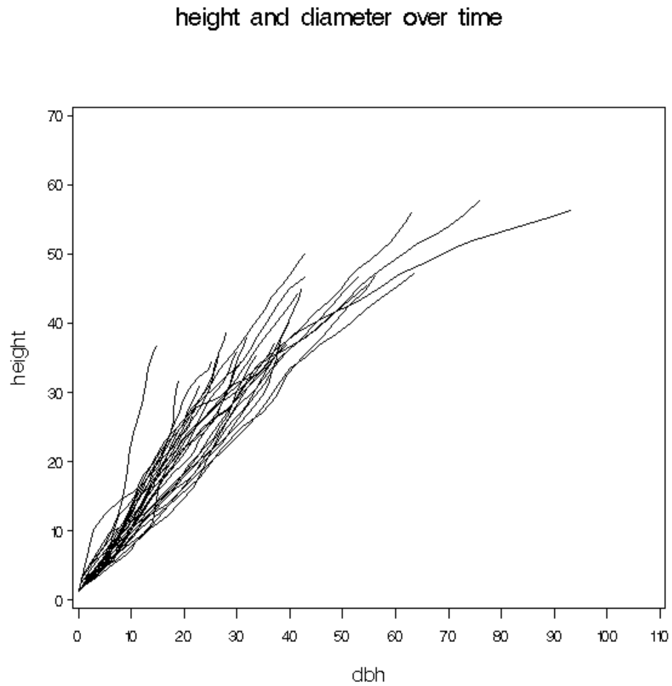
```
* use the conditional means out of MIXED first in plotting
(intermediate or narrow) -- only the last error term as the
residual;
proc plot data=cond4;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
     plot resid*pred='*';
     plot pred*httot='*';
     plot resid*ageper='*';
run;
proc univariate data=cond4 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)
-- all random components in residual;
proc plot data=marg4;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg4 plot normal;
          var resid;
run;

proc sort data=cond4 out=sorted4;
by treeid ageper;
run;
* plot the tree fits using the predicted conditional means;
GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;
AXIS1 LENGTH=4 IN MINOR=NONE ORDER=0 TO 70 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM A=90 R=0 F=SWISS
'predicted height') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 110 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM F=SWISS 'dbh') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=black L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C  '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'predicted height
versus dbh' ;
TITLE3 '  ';
*FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM ' ' ;
*  FOOTNOTE2 ;
```

```
*  FOOTNOTE3 ;
PROC GPLOT DATA = SORTED4;
        PLOT pred * dbhob=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;
```

```
* change which coefficents are allowed to be random;
* fit random coefficients model using PROC MIXED, assuming
correlated errors within trees;

proc mixed data=sorted method=reml maxiter=500 nobound
convf=3E-6;
* default is convh=1E-8;
* can alter this to method=ml and the default of bound also;
title1 'MIXED MODEL 5: varying slopes+correlated errors';
class treeid ageper;
model httot=dbhob dbhsq dbhcu/ solution cl covb
residual outp=cond5 residual outpm=marg5 residual;
random dbhob dbhsq dbhcu/ subject=treeid type=un solution g;
parms (0.2)(0)(0.002)(0)(0)(0)(0.9)(2);
repeated ageper/subject=treeid type=sp(pow)(ageper) r=1,2;
run;
* use the conditional means out of MIXED first in plotting
(intermediate or narrow) -- only the last error term as the
residual;
proc plot data=cond5;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
     plot resid*pred='*';
     plot pred*httot='*';
     plot resid*ageper='*';
run;
proc univariate data=cond5 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged) -- all random components in
residual;
proc plot data=marg5;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg5 plot normal;
          var resid;
run;
proc sort data=cond5 out=sorted5;
by treeid ageper;
run;
```

```
* plot the tree fits using the predicted conditional means;
GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;
* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=4 IN MINOR=NONE ORDER=0 TO 70 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM A=90 R=0 F=SWISS
'predicted height') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 110 BY 10
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.4 CM F=SWISS 'dbh') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=black L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=black L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'predicted height
versus dbh' ;
TITLE3 '  ';
*FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM ' ' ;
*  FOOTNOTE2 ;
*  FOOTNOTE3 ;
PROC GPLOT DATA = SORTED5;
        PLOT pred * dbhob=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;
```

Selected Outputs:



OLS fit -- all trees combined 1

The REG Procedure
Model: MODEL1
Dependent Variable: HTTOT HTTOT

Number of Observations Read	581
Number of Observations Used	581

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	81963	27321	1771.33	<.0001
Error	577	8899.68711	15.42407		
Corrected Total	580	90863			

Root MSE	3.92735	R-Square	0.9021
Dependent Mean	24.53910	Adj R-Sq	0.9015
Coeff Var	16.00445		

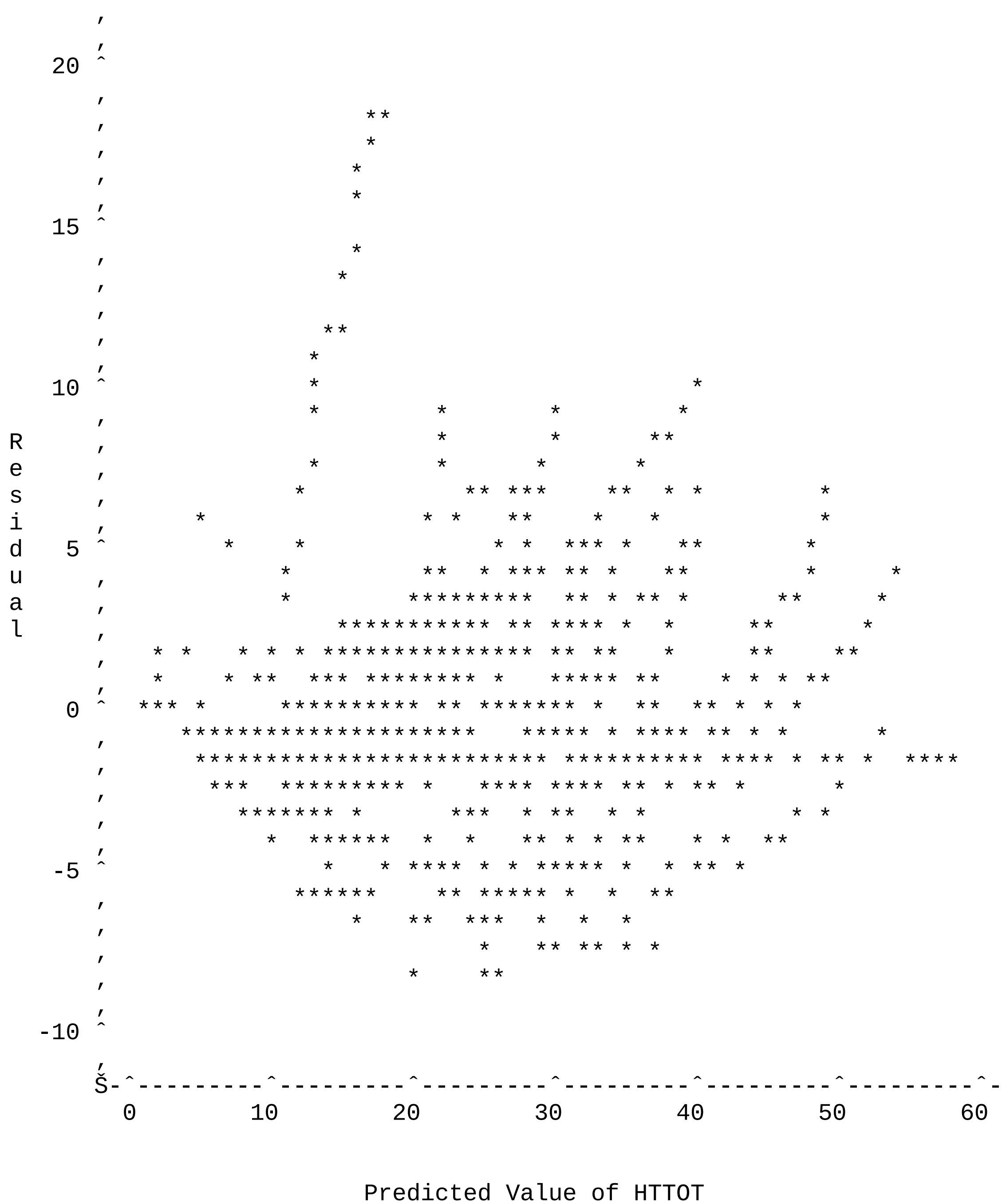
Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	Intercept	1	1.03099	0.57120	1.80	0.0716
DBHOB	DBHOB	1	1.30762	0.06359	20.56	<.0001
dbhsq		1	-0.01089	0.00197	-5.52	<.0001
dbhcu		1	0.00003670	0.00001685	2.18	0.0298

```
                    Plot of yhat*HTTOT.  Symbol used is '*'.

     ,
     ,
  60 ^
     ,                                                         *
     ,                                                       **
     ,                                                      *
     ,                                                     *   * *
     ,                                                    *  *
P 50 ^                                                * * * *  *
r    ,                                               *** * ** *
e    ,                                             ** * **
d    ,                                         * ** * **
i    ,                                       * **** *
c    ,                                      ***** *
t 40 ^                                    *  **   * * *  *
e    ,                                 * * ** ** **** **
d    ,                              * * ******* ** **
     ,                            ** ******  * * *
V    ,                          *  ************
a    ,                          *************
l 30 ^                       ** * ********* ***
u    ,                       ***************
e    ,                     ************ **
     ,                 * ************ **
o    ,                  ** ********* *
f    ,                ************* ****
  20 ^             * ** *******
H    ,              ** ******               *
T    ,           ***********           * ***
T    ,          * ********        ** *
O    ,        **********   * ****
T    ,        ****** **** *
  10 ^       **** **
     ,      ******
     ,     *****   *
     ,    ****   *
     ,    * *
     ,  ****
   0 ^
     ,
     Š-^---------^---------^---------^---------^---------^---------^---------^--
       0        10        20        30        40        50        60

                                   HTTOT
```

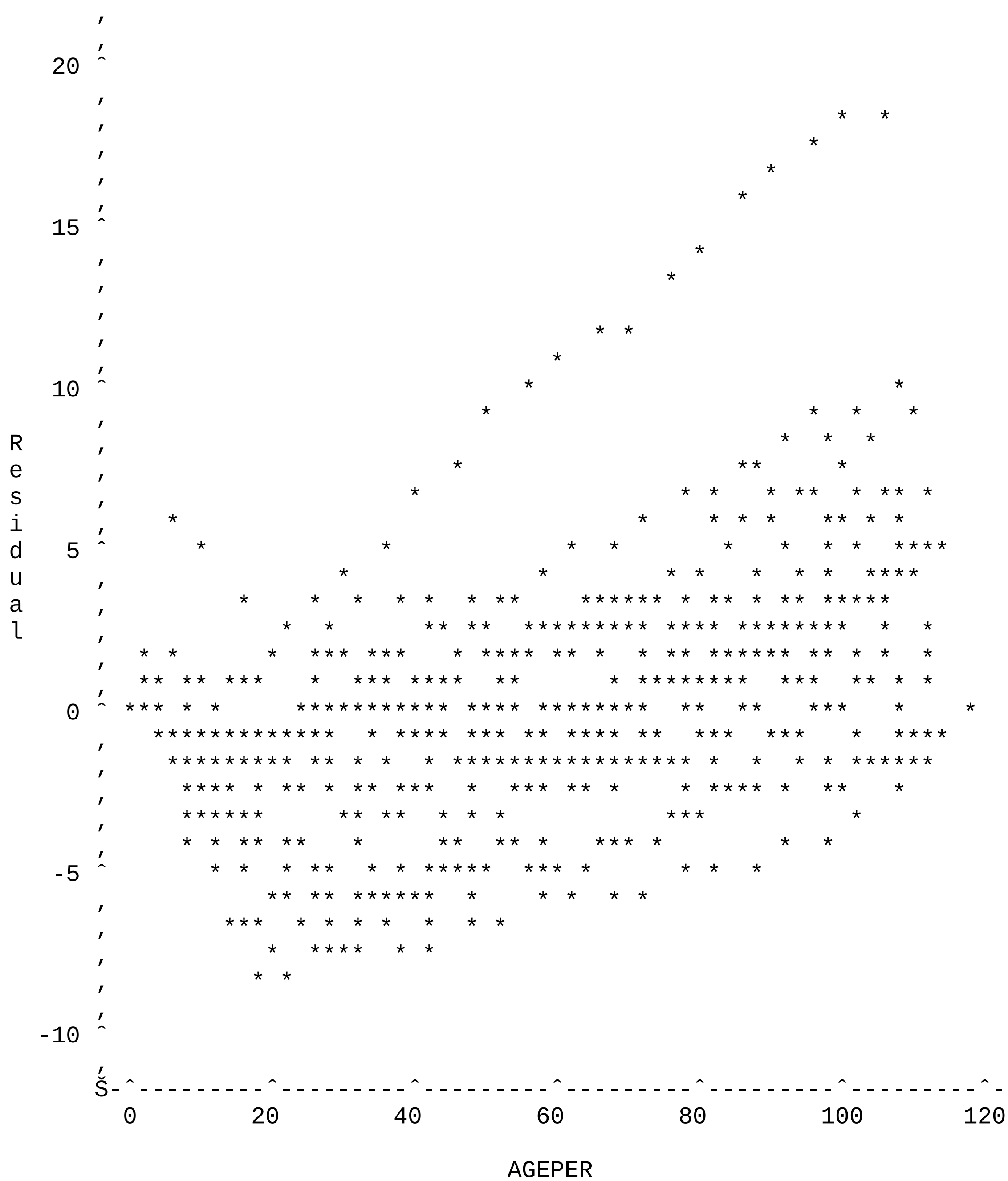
NOTE: 279 obs hidden.

Plot of resid*yhat. Symbol used is '*'.



NOTE: 182 obs hidden.

Plot of resid*AGEPER. Symbol used is '*'.



NOTE: 163 obs hidden.

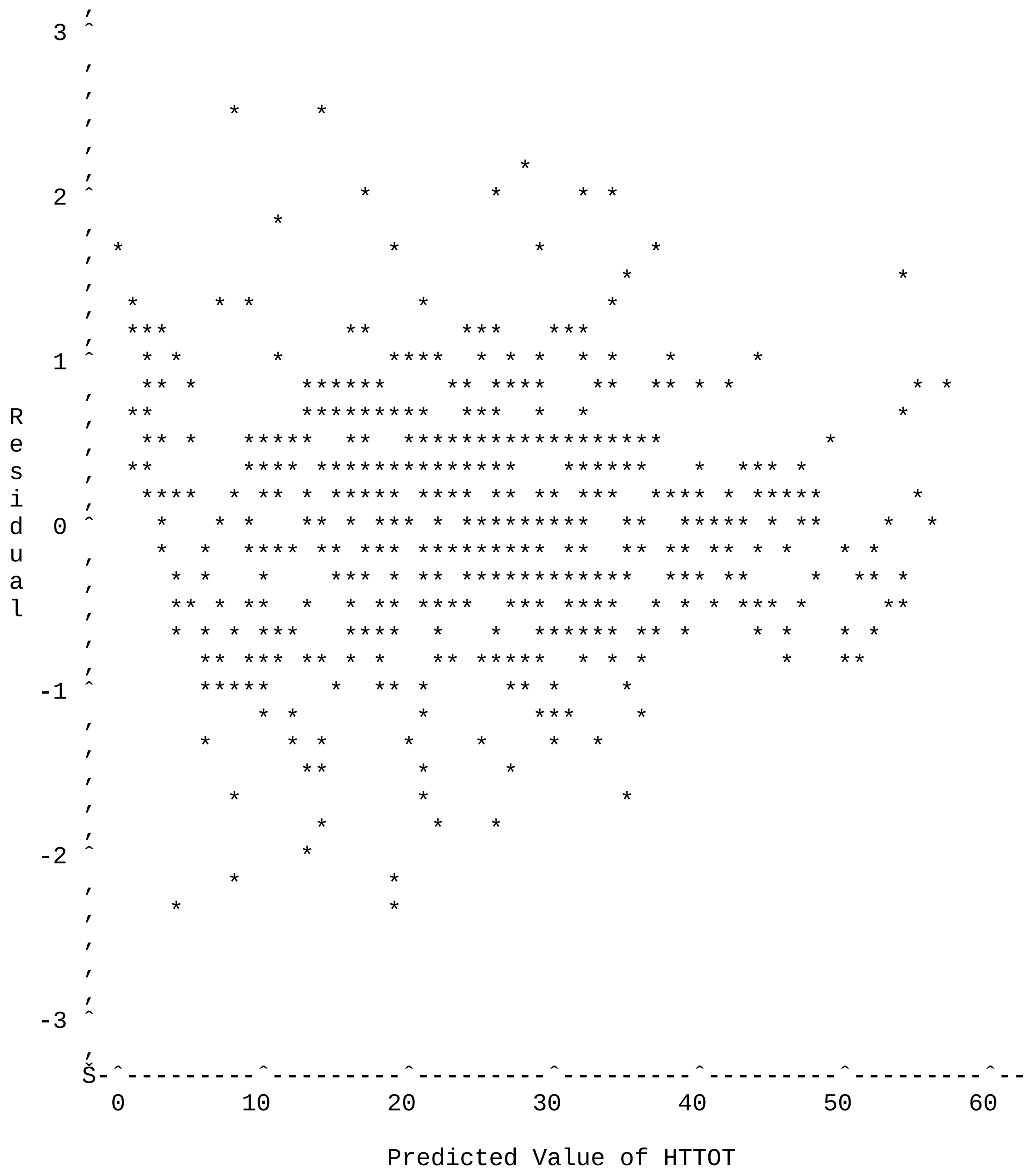
```
                  The UNIVARIATE Procedure
                Variable:  Resid  (Residual)

                  Histogram                       #       Boxplot
19+*                                              2          *
  .*                                              2          *
  .*                                              2          *
  .*                                              1          0
  .*                                              3          0
  .***                                           11          0
  .****                                          16          |
 5+*********                                     33          |
  .*******************                           73          |
  .*****************************                114       +--+--+
  .******************************************   167       *-----*
  .********************                          79       +-----+
  .*************                                 52          |
  .******                                        23          |
-9+*                                              3          0
   ----+----+----+----+----+----+----+----+--
   * may represent up to 4 counts


                       Normal Probability Plot
      19+                                                  *
        |                                                  *
        |                                                **
        |                                                *
        |                                              **
        |                                           ****++++
        |                                        ****++
       5+                                    +****
        |                               +******
        |                         +*******
        |                   ********
        |              ******
        |        *******
        |*********+
     -9+*++++
         +----+----+----+----+----+----+----+----+----+----+
             -2       -1         0        +1        +2
```

OLS by tree --same as if plot was included as dummy variables along with all interactions – regression outputs not shown 5

```
              Plot of yhat*HTTOT.  Symbol used is '*'.

     ,
  60 ˆ
     ,
     ,                                                           *
     ,                                                       ***
     ,                                                      * *
     ,                                                   ***
P 50 ˆ                                                  **
r    ,                                                **
e    ,                                                *
d    ,                                             ***
i    ,                                           ****
c    ,                                         ****
t 40 ˆ                                        ***
e    ,                                       **
d    ,                                    *****
     ,                                  *****
V    ,                                 ****
a    ,                               *****
l 30 ˆ                             *****
u    ,                            *****
e    ,                         *****
     ,                        *****
o    ,                       ****
f    ,                     ****
  20 ˆ                  * ***
H    ,                 ******
T    ,                *****
T    ,              *****
O    ,            ***** *
T    ,           ****
  10 ˆ         *****
     ,       *****
     ,      ****
     ,     ***
     ,  *****
     ,  ****
   0 ˆ  *
     ,
     Š-ˆ---------ˆ---------ˆ---------ˆ---------ˆ---------ˆ---------ˆ--
       0        10        20        30        40        50        60

                                   HTTOT

NOTE: 447 obs hidden.
```

OLS by tree -- same as if plot was included as dummy variables along with all interactions

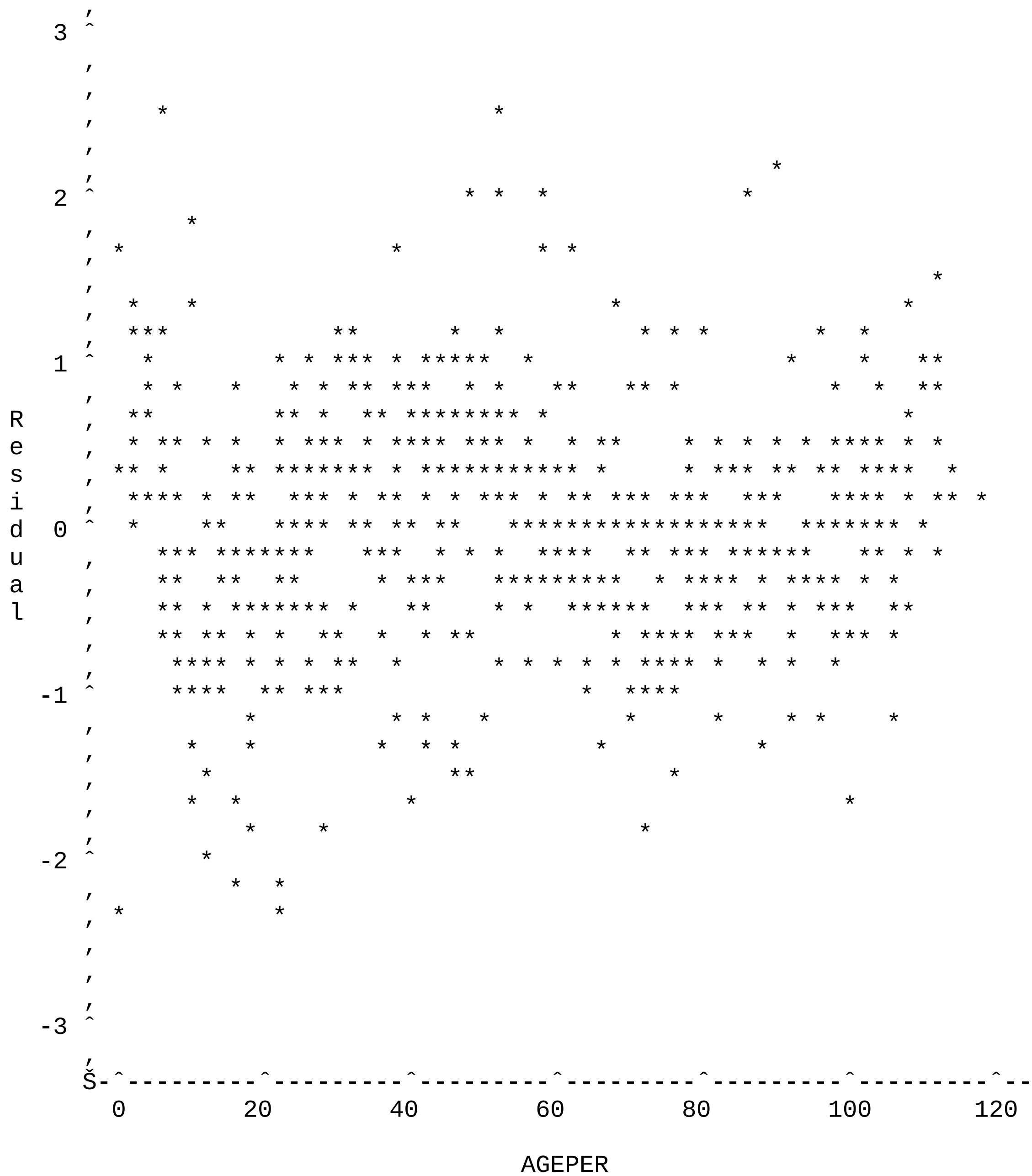
Plot of resid*yhat. Symbol used is '*'.



NOTE: 191 obs hidden.

OLS by tree -- same as if plot was included as dummy variables along with all interactions 7

Plot of resid*AGEPER. Symbol used is '*'.



NOTE: 158 obs hidden.

OLS by tree -- same as if plot was included as dummy variables along with all interactions -- variability in the coefficients 8

The CORR Procedure

```
4 With Variables:    Intercept DBHOB     dbhsq     dbhcu
4      Variables:    Intercept DBHOB     dbhsq     dbhcu
```

Covariance Matrix, DF = 26

		Intercept	DBHOB
Intercept	Intercept	6.344108582	-2.479565627
DBHOB	DBHOB	-2.479565627	1.274488080
dbhsq		0.315007821	-0.168353919
dbhcu		-0.009844916	0.005355016

Covariance Matrix, DF = 26

		dbhsq	dbhcu
Intercept	Intercept	0.315007821	-0.009844916
DBHOB	DBHOB	-0.168353919	0.005355016
dbhsq		0.024295373	-0.000793211
dbhcu		-0.000793211	0.000026288

Simple Statistics

Variable	N	Mean	Std Dev	Sum
Intercept	27	1.21833	2.51875	32.89488
DBHOB	27	0.92548	1.12893	24.98807
dbhsq	27	0.01801	0.15587	0.48621
dbhcu	27	-0.0005833	0.00513	-0.01575

Simple Statistics

Variable	Minimum	Maximum	Label
Intercept	-3.42034	11.38108	Intercept
DBHOB	-4.17396	2.27374	DBHOB
dbhsq	-0.11514	0.77481	
dbhcu	-0.02536	0.00584	

The Mixed Procedure

Model Information

Data Set	WORK.SORTED
Dependent Variable	HTTOT
Covariance Structure	Diagonal
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Residual

Dimensions

Covariance Parameters	1
Columns in X	4
Columns in Z	0
Subjects	1
Max Obs Per Subject	581

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate
Residual	15.4241

Fit Statistics

-2 Res Log Likelihood	3277.3
AIC (smaller is better)	3279.3
AICC (smaller is better)	3279.3
BIC (smaller is better)	3283.6

MIXED MODEL: as OLS 11

The Mixed Procedure

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	1.0310	0.5712	577	1.80	0.0716	0.05
DBHOB	1.3076	0.06359	577	20.56	<.0001	0.05
dbhsq	-0.01089	0.001974	577	-5.52	<.0001	0.05
dbhcu	0.000037	0.000017	577	2.18	0.0298	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-0.09090	2.1529
DBHOB	1.1827	1.4325
dbhsq	-0.01477	-0.00702
dbhcu	3.605E-6	0.000070

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.3263	-0.03227	0.000851	-6.26E-6
2	DBHOB	-0.03227	0.004044	-0.00012	9.477E-7
3	dbhsq	0.000851	-0.00012	3.895E-6	-3.25E-8
4	dbhcu	-6.26E-6	9.477E-7	-3.25E-8	2.84E-10

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	577	422.84	<.0001
dbhsq	1	577	30.48	<.0001
dbhcu	1	577	4.74	0.0298

The Mixed Procedure

Model Information

Data Set	WORK.SORTED
Dependent Variable	HTTOT
Covariance Structure	Spatial Power
Subject Effect	treeid
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Between-Within

Class Level Information

Class	Levels	Values
treeid	27	312 320 401 407 501 515 518 525 529 603 615 621 626 704 708 709 710 722 802 804 806 810 812 901 906 907 916
AGEPER	116	0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 117

Dimensions

Covariance Parameters	2
Columns in X	4
Columns in Z	0
Subjects	27
Max Obs Per Subject	24

The Mixed Procedure

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	3277.26038865	
1	2	1995.11325803	115623.71129
2	1	1715.25639578	6.31695140
3	1	1578.69600667	0.55199678
4	1	1527.11932293	0.08447062
5	1	1514.43679647	0.00064681
6	1	1514.29928912	0.00000512
7	1	1514.29812109	0.00000000

Convergence criteria met.

Estimated R Matrix for treeid 312

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	22.5047	22.1870	21.8738	21.5650	21.2605	20.9603
2	22.1870	22.5047	22.1870	21.8738	21.5650	21.2605
3	21.8738	22.1870	22.5047	22.1870	21.8738	21.5650
4	21.5650	21.8738	22.1870	22.5047	22.1870	21.8738
5	21.2605	21.5650	21.8738	22.1870	22.5047	22.1870
6	20.9603	21.2605	21.5650	21.8738	22.1870	22.5047
7	20.6644	20.9603	21.2605	21.5650	21.8738	22.1870
8	20.3727	20.6644	20.9603	21.2605	21.5650	21.8738
9	20.0850	20.3727	20.6644	20.9603	21.2605	21.5650
10	19.8015	20.0850	20.3727	20.6644	20.9603	21.2605
11	19.5219	19.8015	20.0850	20.3727	20.6644	20.9603
12	19.2463	19.5219	19.8015	20.0850	20.3727	20.6644
13	18.9745	19.2463	19.5219	19.8015	20.0850	20.3727
14	18.7067	18.9745	19.2463	19.5219	19.8015	20.0850
15	18.4426	18.7067	18.9745	19.2463	19.5219	19.8015
16	18.1822	18.4426	18.7067	18.9745	19.2463	19.5219
17	17.9255	18.1822	18.4426	18.7067	18.9745	19.2463
18	17.6724	17.9255	18.1822	18.4426	18.7067	18.9745

ETC…

The Mixed Procedure

Estimated R Matrix for treeid 320

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	22.5047	22.1870	21.8738	21.5650	21.2605	20.9603
2	22.1870	22.5047	22.1870	21.8738	21.5650	21.2605
3	21.8738	22.1870	22.5047	22.1870	21.8738	21.5650

Estimated R Matrix for treeid 320

Row	Col7	Col8	Col9	Col10	Col11	Col12
1	20.6644	20.3727	20.0850	19.8015	19.5219	19.2463
2	20.9603	20.6644	20.3727	20.0850	19.8015	19.5219
3	21.2605	20.9603	20.6644	20.3727	20.0850	19.8015

Estimated R Matrix for treeid 320

Row	Col13	Col14	Col15	Col16	Col17	Col18
1	18.9745	18.7067	18.4426	18.1822	17.9255	17.6724
2	19.2463	18.9745	18.7067	18.4426	18.1822	17.9255
3	19.5219	19.2463	18.9745	18.7067	18.4426	18.1822

Estimated R Matrix for treeid 320

Row	Col19	Col20	Col21	Col22	Col23
1	17.4229	17.1769	16.9344	16.6953	16.4596
2	17.6724	17.4229	17.1769	16.9344	16.6953
3	17.9255	17.6724	17.4229	17.1769	16.9344

Etc..

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
SP(POW)	treeid	0.9972
Residual		22.5047

Fit Statistics

-2 Res Log Likelihood	**1514.3**
AIC (smaller is better)	1518.3
AICC (smaller is better)	1518.3
BIC (smaller is better)	1520.9

Null Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
1	1762.96	<.0001

The Mixed Procedure

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	4.1909	0.9023	26	4.64	<.0001	0.05
DBHOB	0.8974	0.03302	551	27.18	<.0001	0.05
dbhsq	0.001229	0.001127	551	1.09	0.2760	0.05
dbhcu	-0.00004	9.812E-6	551	-4.55	<.0001	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	2.3361	6.0457
DBHOB	0.8326	0.9623
dbhsq	-0.00098	0.003442
dbhcu	-0.00006	-0.00003

Covariance Matrix for Fixed Effects

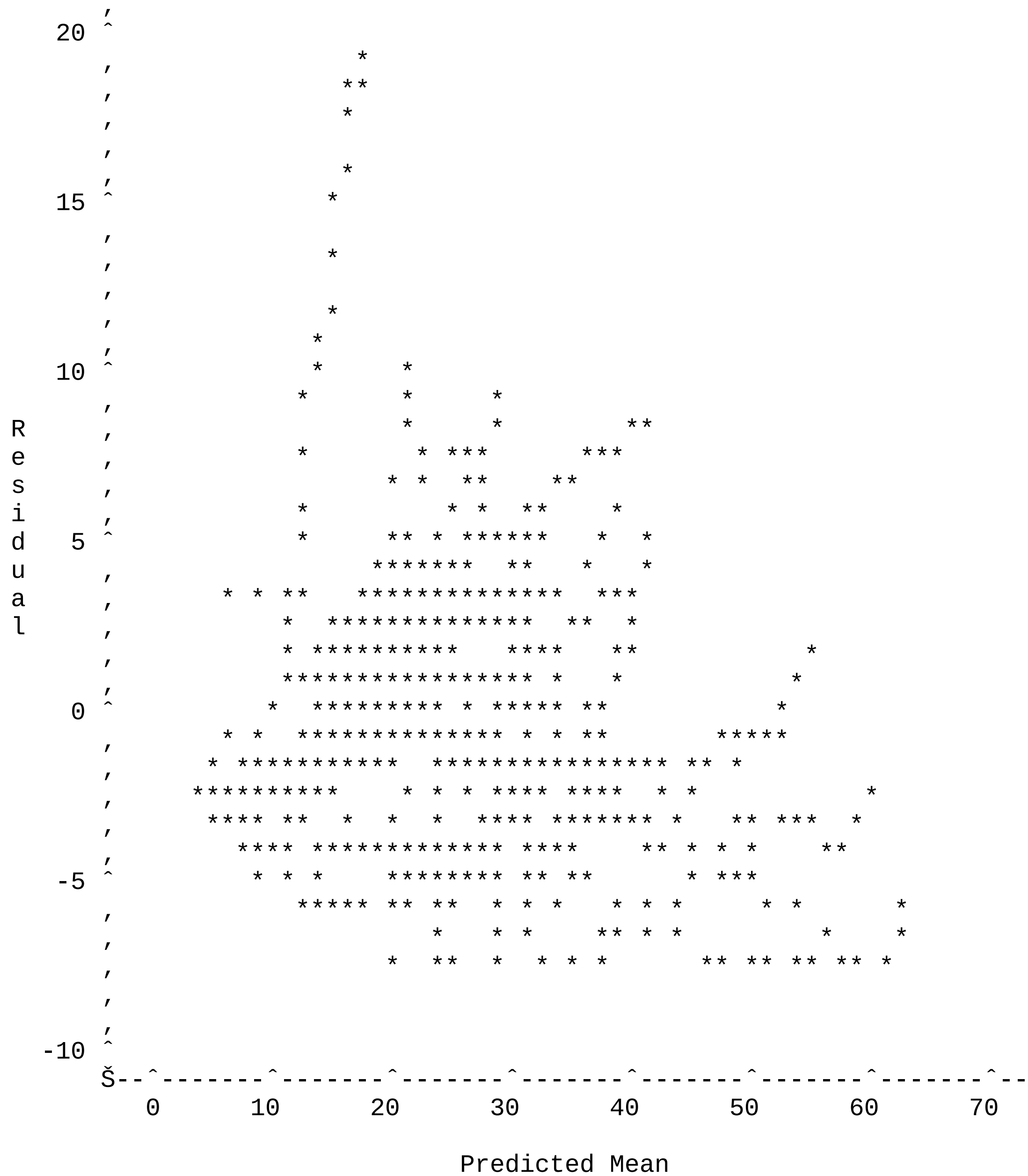
Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.8142	-0.00492	0.000020	-1.22E-8
2	DBHOB	-0.00492	0.001090	-0.00003	2.343E-7
3	dbhsq	0.000020	-0.00003	1.27E-6	-1.05E-8
4	dbhcu	-1.22E-8	2.343E-7	-1.05E-8	9.63E-11

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	551	738.66	<.0001
dbhsq	1	551	1.19	0.2760
dbhcu	1	551	20.71	<.0001

MIXED MODEL 1: correlated error terms
marginal residuals (all random components)
and predicted values using fixed effects only

Plot of Resid*Pred. Symbol used is '*'.



NOTE: 227 obs hidden.

```
              MIXED MODEL 1: correlated error terms               28
            marginal residuals (all random components)
          and predicted values using fixed effects only

                      Tests for Normality

Test                  --Statistic---    -----p Value------

Shapiro-Wilk          W     0.944909    Pr < W     <0.0001
Kolmogorov-Smirnov    D     0.071241    Pr > D     <0.0100
Cramer-von Mises      W-Sq  0.764929    Pr > W-Sq  <0.0050
Anderson-Darling      A-Sq    4.8009    Pr > A-Sq  <0.0050


                         Normal Probability Plot
     19+                                                *
       |                                               **
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       |                                  +****
       |                             +******
       |                        +******
       |                    ******
       |             ********
       |       *******+
    -7+**********++
       +----+----+----+----+----+----+----+----+----+----+
           -2        -1         0        +1        +2
```

MIXED MODEL 2: varying slopes 31

```
                      The Mixed Procedure

                       Model Information

  Data Set                     WORK.SORTED
  Dependent Variable           HTTOT
  Covariance Structure         Unstructured
  Subject Effect               treeid
  Estimation Method            REML
  Residual Variance Method     Profile
  Fixed Effects SE Method      Model-Based
  Degrees of Freedom Method    Containment
```

Class Level Information

Class	Levels	Values
treeid	27	312 320 401 407 501 515 518 525 529 603 615 621 626 704 708 709 710 722 802 804 806 810 812 901 906 907 916

Dimensions

Covariance Parameters	4
Columns in X	4
Columns in Z Per Subject	2
Subjects	27
Max Obs Per Subject	24

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Parameter Search

CovP1	CovP2	CovP3	CovP4	Variance	Res Log Like
3.0000	0	0.6000	16.0000	2.5516	-1205.8210

The Mixed Procedure

Parameter Search

-2 Res Log Like
2411.6419

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
1	2	2376.13976012	0.02661875
2	1	2353.03529595	0.01585383
3	1	2339.65467282	0.00823839
4	1	2332.88984409	0.00342888
5	1	2330.17237296	0.00094433
6	1	2329.46589495	0.00011070
7	1	2329.38989026	0.00000208
8	1	2329.38855558	0.00000000

Convergence criteria met.

Estimated G Matrix

Row	Effect	treeid	Col1	Col2
1	Intercept	312	5.8658	-0.6639
2	DBHOB	312	-0.6639	0.1590

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	treeid	5.8658
UN(2,1)	treeid	-0.6639
UN(2,2)	treeid	0.1590
Residual		2.0185

The Mixed Procedure

Fit Statistics

-2 Res Log Likelihood	**2329.4**
AIC (smaller is better)	2337.4
AICC (smaller is better)	2337.5
BIC (smaller is better)	2342.6

PARMS Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
3	82.25	<.0001

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	0.6860	0.5186	26	1.32	0.1974	0.05
DBHOB	1.0610	0.08155	26	13.01	<.0001	0.05
dbhsq	0.003438	0.000941	525	3.65	0.0003	0.05
dbhcu	-0.00006	8.02E-6	525	-7.60	<.0001	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	-0.3799	1.7519
DBHOB	0.8933	1.2286
dbhsq	0.001588	0.005287
dbhcu	-0.00008	-0.00005

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.2689	-0.03004	0.000152	-1.09E-6
2	DBHOB	-0.03004	0.006650	-0.00002	1.899E-7
3	dbhsq	0.000152	-0.00002	8.862E-7	-7.23E-9
4	dbhcu	-1.09E-6	1.899E-7	-7.23E-9	6.43E-11

The Mixed Procedure

Solution for Random Effects

Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	312	0.5250	0.7866	525	0.67	0.5048
DBHOB	312	-0.2432	0.07864	525	-3.09	0.0021
Intercept	320	2.5236	0.7739	525	3.26	0.0012
DBHOB	320	-0.3047	0.07889	525	-3.86	0.0001
Intercept	401	-1.5420	0.8096	525	-1.90	0.0574
DBHOB	401	-0.1537	0.07888	525	-1.95	0.0518
Intercept	407	0.6770	0.7781	525	0.87	0.3847
DBHOB	407	0.05029	0.08027	525	0.63	0.5312
Intercept	501	-1.6636	0.7397	525	-2.25	0.0249
DBHOB	501	-0.08507	0.08130	525	-1.05	0.2959
Intercept	515	-0.1059	0.9008	525	-0.12	0.9065
DBHOB	515	0.1182	0.09311	525	1.27	0.2049
Intercept	518	2.4146	0.8568	525	2.82	0.0050
DBHOB	518	0.1055	0.08877	525	1.19	0.2354
Intercept	525	4.3128	0.7324	525	5.89	<.0001
DBHOB	525	-0.2905	0.07952	525	-3.65	0.0003
Intercept	529	3.3192	0.7069	525	4.70	<.0001
DBHOB	529	-0.1431	0.08220	525	-1.74	0.0822

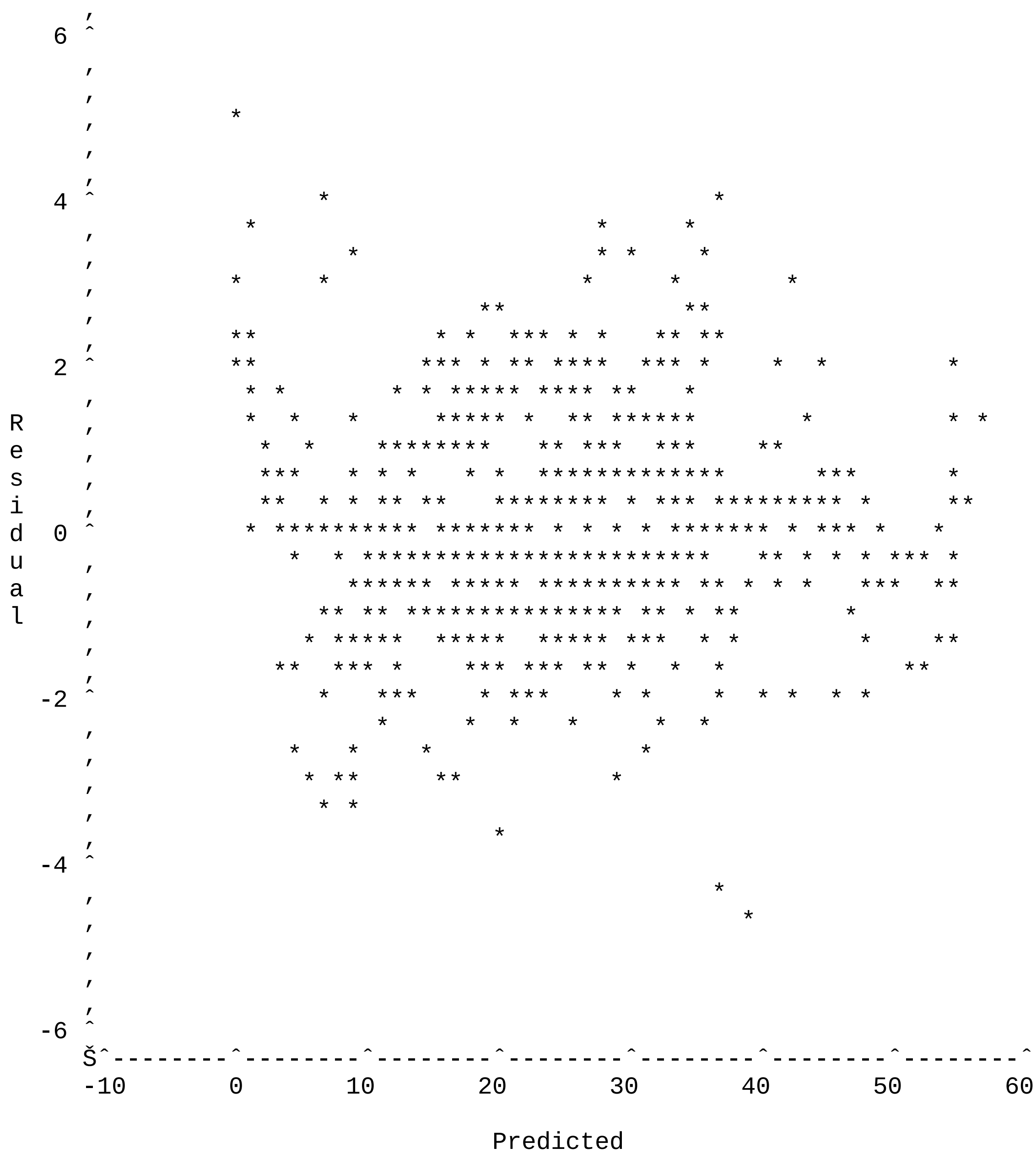
Etc.

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	26	169.26	<.0001
dbhsq	1	525	13.34	0.0003
dbhcu	1	525	57.77	<.0001

MIXED MODEL 2: varying slopes
conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Resid*Pred. Symbol used is '*'.



NOTE: 212 obs hidden.

MIXED MODEL 2: varying slopes 37

conditional residuals (only white noise) and predicted values

using fixed+random effects

Plot of Pred*HTTOT. Symbol used is '*'.

```
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  60 ˆ
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  40 ˆ                                   *   *****
P    ,                                      ***
r    ,                                  ********
e    ,                                  *******
d    ,                                 ******
i    ,                             *******
c 30 ˆ                             ***** *
t    ,                           *******
e    ,                          ******
d    ,                        ******
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  20 ˆ                 ********
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   0 ˆ   ** *
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       0        10        20        30        40        50        60

                                  HTTOT
```

NOTE: 407 obs hidden.

conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Resid*AGEPER. Symbol used is '*'.

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R     ,  *  **     **      *   *  * *  * ** **  *  *   * *    ***
e     ,  *  * *   *   ***** ***** *  * * **     * *       ** *
s     ,   *** *  * *  **** **** * **   ** *   * *** ******  ****
i     ,  ** * * ** **** * **  * ** *** ** * * **** **  ***** ** *
d   0 ^ ** ****** **** ** * *  *  * ***  **** ***** ****   *  *
u     ,    * * *** **** **   ** **  ** **** ************ ** * * *
a     ,      *******  * * ** *  * ****** ** **** **** ***  * **
l     ,     **** *************** *** ***  * ** * *      ** *    *
      ,  ** * ** *** * **  * * *  ** *  * *** ***     *  **
      , **       * *  * * * *  *** ** ** **  *   *  *
   -2 ^     * * *         ** **** * *      *              * *  * *
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      ,   *  *
      ,            *
   -4 ^
      ,                                                      *
      ,                                                        *
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      ,
   -6 ^
      Š-^---------^---------^---------^---------^---------^---------^--
        0        20        40        60        80       100       120

                                    AGEPER
```

NOTE: 168 obs hidden.

conditional residuals (only white noise) and predicted values
using fixed+random effects

The UNIVARIATE Procedure
Variable: Resid (Residual)

Test	--Statistic---		-----p Value------	
Shapiro-Wilk	W	0.992633	Pr < W	0.0058
Kolmogorov-Smirnov	D	0.041165	Pr > D	0.0178
Cramer-von Mises	W-Sq	0.251054	Pr > W-Sq	<0.0050
Anderson-Darling	A-Sq	1.451449	Pr > A-Sq	<0.0050

Normal Probability Plot

```
 5.25+                                                  *
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 4.25+
     |                                                ***
 3.25+                                              ***++
     |                                           ****+
 2.25+                                       *****
     |                                    ****
 1.25+                                 ****
     |                             +****
 0.25+                         +*****
     |                      *****
-0.75+                 ******
     |             *****
-1.75+         *****
     |      ***++
-2.75+   +**+
     |****
-3.75+*
     |*
-4.75+*
      +----+----+----+----+----+----+----+----+----+----+
          -2        -1         0        +1        +2
```

MIXED MODEL 2: varying slopes
marginal residuals (all random components)
and predicted values using fixed effects only

```
                    Plot of Resid*Pred.  Symbol used is '*'.

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           ,   ** ************* *****
     R   0 ˆ  ************************** *
     e     ,     ************************ **
     s     ,      *******  ****** ****   **
     i     ,        ********** * ***** ***
     d     ,           *  * *** *** *****
     u     ,              *  **************
     a     ,                 **  **** *** * **  *
     l -10 ˆ                     * * **    ****** *  **
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       -30 ˆ
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              0          20          40          60          80

                               Predicted Mean

NOTE: 310 obs hidden.
```

```
              MIXED MODEL 2: varying slopes                   44
        marginal residuals (all random components)
      and predicted values using fixed effects only

                 The UNIVARIATE Procedure
               Variable:  Resid  (Residual)

                    Tests for Normality

Test                  --Statistic---    -----p Value------
Shapiro-Wilk          W     0.946347    Pr < W     <0.0001
Kolmogorov-Smirnov    D     0.100642    Pr > D     <0.0100
Cramer-von Mises      W-Sq  1.788165    Pr > W-Sq  <0.0050
Anderson-Darling      A-Sq  10.15629    Pr > A-Sq  <0.0050

                      Normal Probability Plot
     19+                                                  *
       |                                                 *
       |                                                *
       |                                              **+++
       |                                             **+
       |                                          ++**
       |                                      ++++***
       |                                   +++ ****
       |                                ********
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       |                       *****+
     -3+                    ***++
       |                 ****+
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       |            ***
       |         +***
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       |++  ***
       |  **
       | **
       |*
    -25+*
        +----+----+----+----+----+----+----+----+----+----+
            -2        -1         0        +1        +2
```

The Mixed Procedure

Model Information

Data Set	WORK.SORTED
Dependent Variable	HTTOT
Covariance Structures	Unstructured, Spatial Power
Subject Effects	treeid, treeid
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
treeid	27	312 320 401 407 501 515 518 525 529 603 615 621 626 704 708 709 710 722 802 804 806 810 812 901 906 907 916
AGEPER	116	0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 117

Dimensions

Covariance Parameters	5
Columns in X	4
Columns in Z Per Subject	2
Subjects	27
Max Obs Per Subject	24

MIXED MODEL 3: varying slopes+correlated errors 49

The Mixed Procedure

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Parameter Search

CovP1	CovP2	CovP3	CovP4	CovP5	Variance
6.0000	0	1.0000	0.9000	2.0000	1.1903

Parameter Search

Res Log Like	-2 Res Log Like
-898.0979	1796.1959

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
1	2	1634.59931136	15057.777214
2	1	1575.40158832	3817.1808556
3	1	1557.08168144	2251.0759403
4	2	1539.90882655	1111.9432724
5	2	1524.57393090	519.96782417
6	2	1511.88013487	457.18307096
7	2	1502.67226664	908.14582087
8	2	1497.24346188	1217.4356071
9	1	1495.15186320	4212.1204254
10	1	1493.29469366	272.49487609
11	3	1465.10583755	0.00227963
12	1	1464.66401402	0.00005742
13	1	1464.65221821	0.00000009
14	1	1464.65220038	0.00000000

Convergence criteria met.

MIXED MODEL 3: varying slopes+correlated errors 50

The Mixed Procedure

Estimated R Matrix for treeid 312

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	8.8144	8.5385	8.2714	8.0125	7.7618	7.5189
2	8.5385	8.8144	8.5385	8.2714	8.0125	7.7618
3	8.2714	8.5385	8.8144	8.5385	8.2714	8.0125
4	8.0125	8.2714	8.5385	8.8144	8.5385	8.2714
5	7.7618	8.0125	8.2714	8.5385	8.8144	8.5385
6	7.5189	7.7618	8.0125	8.2714	8.5385	8.8144
7	7.2836	7.5189	7.7618	8.0125	8.2714	8.5385
8	7.0557	7.2836	7.5189	7.7618	8.0125	8.2714
9	6.8349	7.0557	7.2836	7.5189	7.7618	8.0125
10	6.6210	6.8349	7.0557	7.2836	7.5189	7.7618
11	6.4138	6.6210	6.8349	7.0557	7.2836	7.5189

Etc

Estimated G Matrix

Row	Effect	treeid	Col1	Col2
1	Intercept	312		0.2219
2	DBHOB	312	0.2219	0.03203

Estimated R Matrix for treeid 320

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	8.8144	8.5385	8.2714	8.0125	7.7618	7.5189
2	8.5385	8.8144	8.5385	8.2714	8.0125	7.7618
3	8.2714	8.5385	8.8144	8.5385	8.2714	8.0125
4	8.0125	8.2714	8.5385	8.8144	8.5385	8.2714
5	7.7618	8.0125	8.2714	8.5385	8.8144	8.5385

ETC…

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	treeid	0
UN(2,1)	treeid	0.2219
UN(2,2)	treeid	0.03203
SP(POW)	treeid	0.9937
Residual		8.8144

Fit Statistics

-2 Res Log Likelihood	1464.7
AIC (smaller is better)	1472.7
AICC (smaller is better)	1472.7
BIC (smaller is better)	1477.8

MIXED MODEL 3: varying slopes+correlated errors 56

The Mixed Procedure

PARMS Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
3	331.54	<.0001

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	3.1639	0.5699	26	5.55	<.0001	0.05
DBHOB	0.9116	0.04625	26	19.71	<.0001	0.05
dbhsq	0.003328	0.001104	525	3.01	0.0027	0.05
dbhcu	-0.00005	9.829E-6	525	-5.26	<.0001	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	1.9923	4.3354
DBHOB	0.8166	1.0067
dbhsq	0.001158	0.005498
dbhcu	-0.00007	-0.00003

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.3248	0.003710	0.000014	-6.45E-8
2	DBHOB	0.003710	0.002139	-0.00003	2.142E-7
3	dbhsq	0.000014	-0.00003	1.22E-6	-1.01E-8
4	dbhcu	-6.45E-8	2.142E-7	-1.01E-8	9.66E-11

Solution for Random Effects

Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	312	-0.6938	0	525	-Infty	<.0001
DBHOB	312	-0.1303	0.04737	525	-2.75	0.0062
Intercept	320	-1.6002	0	525	-Infty	<.0001
DBHOB	320	-0.1970	0.04898	525	-4.02	<.0001

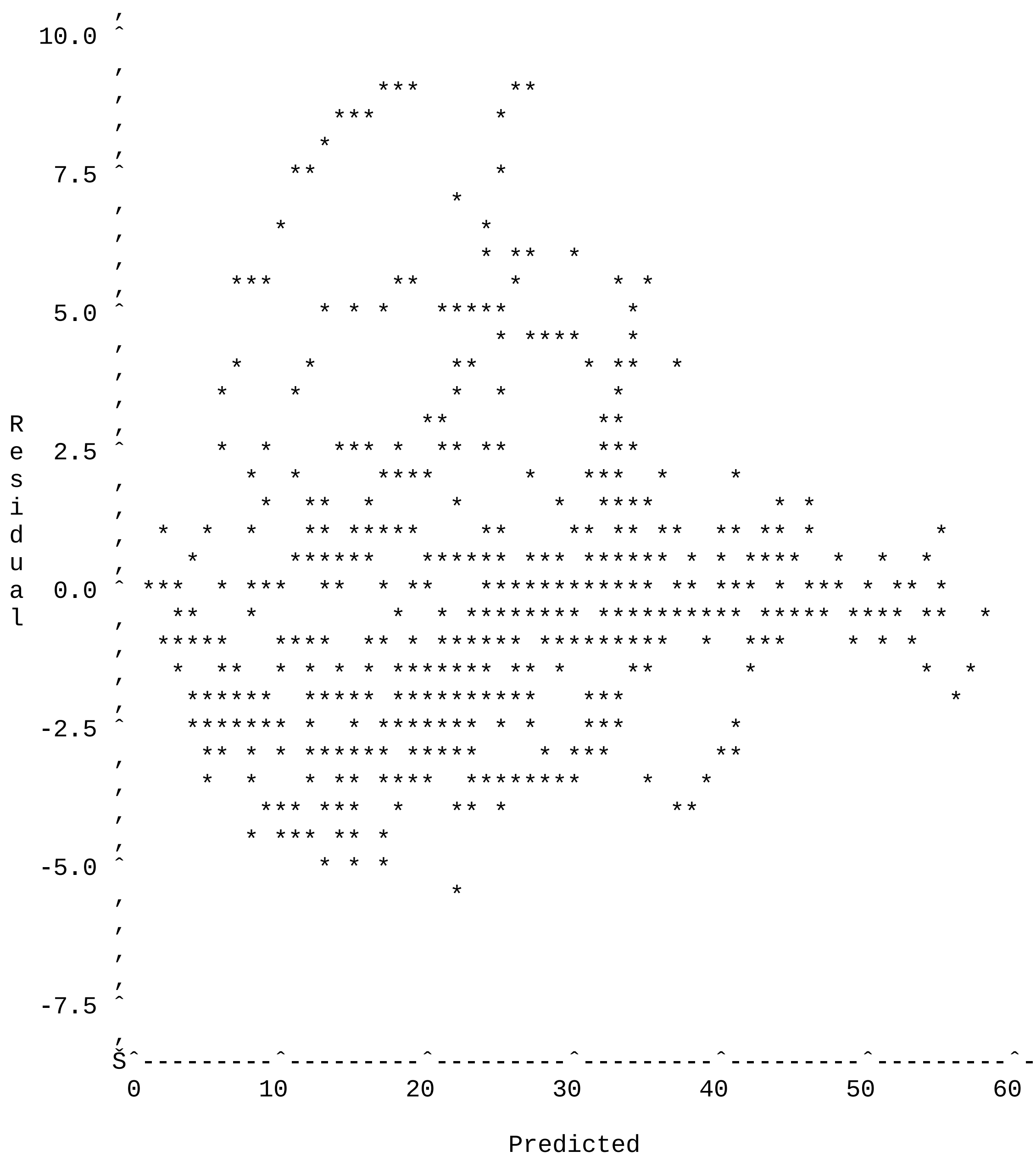
Etc..

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	26	388.47	<.0001
dbhsq	1	525	9.08	0.0027
dbhcu	1	525	27.64	<.0001

conditional residuals (only white noise) and predicted values
using fixed+random effects

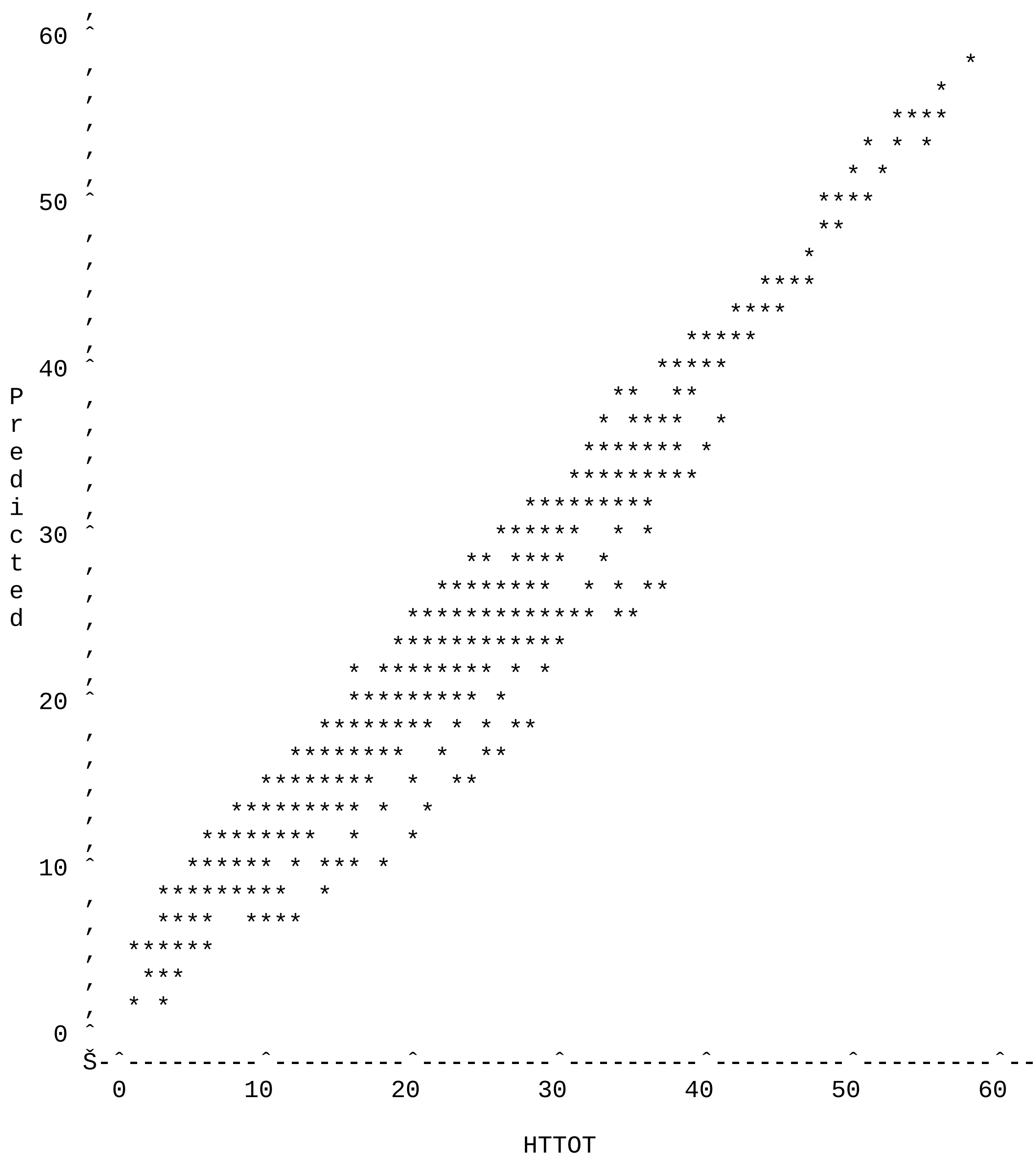
Plot of Resid*Pred. Symbol used is '*'.



NOTE: 191 obs hidden.

MIXED MODEL 3: varying slopes+correlated errors 60
conditional residuals (only white noise) and predicted values
using fixed+random effects

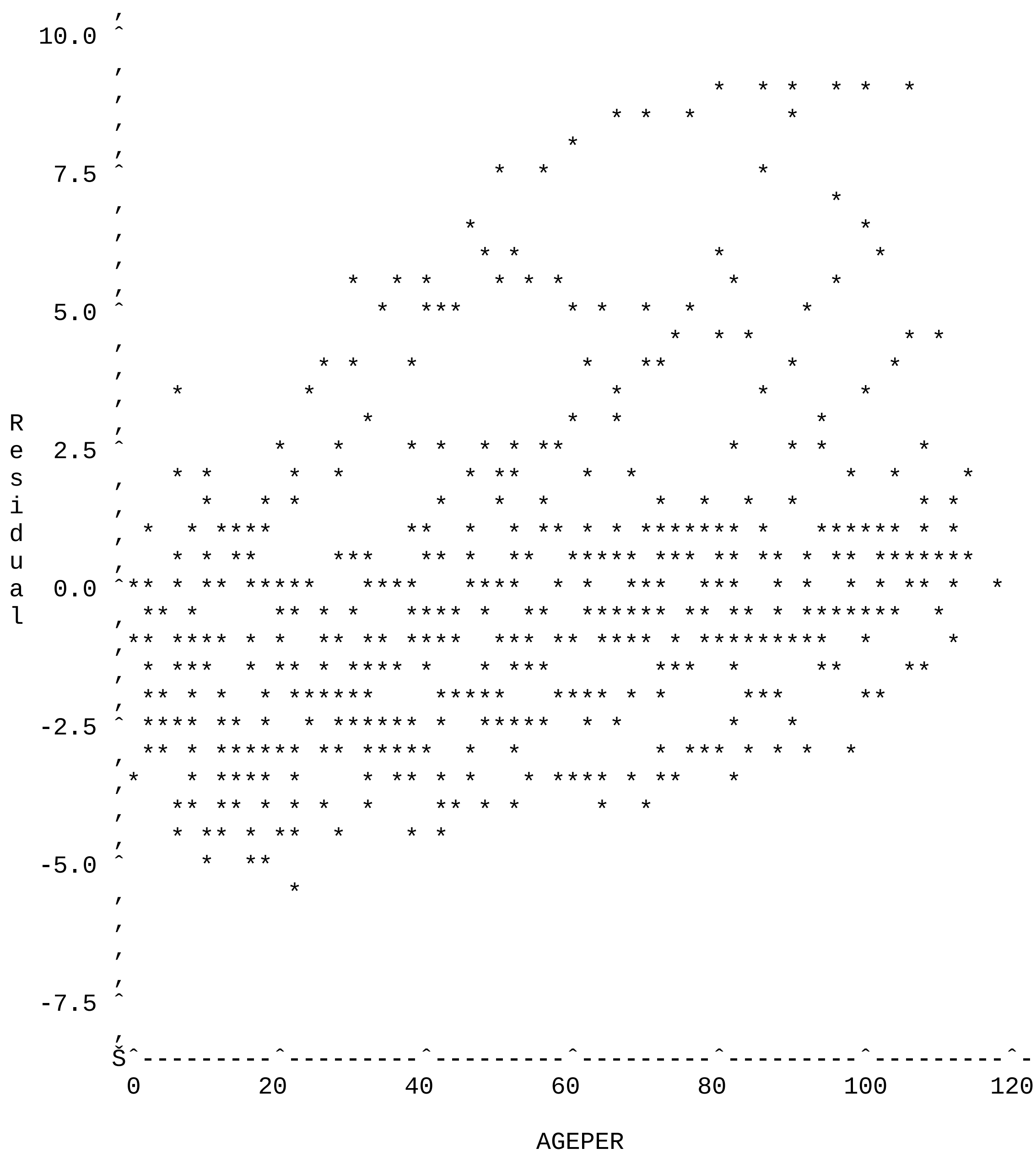
Plot of Pred*HTTOT. Symbol used is '*'.



NOTE: 339 obs hidden.

conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Resid*AGEPER. Symbol used is '*'.



NOTE: 167 obs hidden.

MIXED MODEL 3: varying slopes+correlated errors 62
conditional residuals (only white noise) and predicted values
using fixed+random effects

The UNIVARIATE Procedure
Variable: Resid (Residual)

Tests for Normality

Test	--Statistic---		-----p Value------	
Shapiro-Wilk	W	0.925989	Pr < W	<0.0001
Kolmogorov-Smirnov	D	0.106991	Pr > D	<0.0100
Cramer-von Mises	W-Sq	1.635888	Pr > W-Sq	<0.0050
Anderson-Darling	A-Sq	10.58314	Pr > A-Sq	<0.0050

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                      Normal Probability Plot
    9.5+                                                  *
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    6.5+                                            **   ++
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       |                                  ++**
       |                              +++****
    0.5+                          ++******
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       |                   *****
   -2.5+              *****
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       | *******+++
   -5.5+*   ++++
        +----+----+----+----+----+----+----+----+----+----+
            -2        -1         0        +1        +2
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marginal residuals (all random components)
and predicted values using fixed effects only

Plot of Resid*Pred. Symbol used is '*'.

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              0          20          40          60          80

                               Predicted Mean
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NOTE: 333 obs hidden.

```
MIXED MODEL 3: varying slopes+correlated errors          66
    marginal residuals (all random components)
  and predicted values using fixed effects only

             The UNIVARIATE Procedure
           Variable:  Resid  (Residual)

                  Normal Probability Plot
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        -2        -1         0        +1        +2
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MIXED MODEL 4: varying slopes+correlated errors 70

The Mixed Procedure

Model Information

Data Set	WORK.SORTED
Dependent Variable	HTTOT
Covariance Structures	Unstructured, Spatial Power
Subject Effects	treeid, treeid
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
treeid	27	312 320 401 407 501 515 518 525 529 603 615 621 626 704 708 709 710 722 802 804 806 810 812 901 906 907 916
AGEPER	116	0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 117

Dimensions

Covariance Parameters	5
Columns in X	4
Columns in Z Per Subject	2
Subjects	27
Max Obs Per Subject	24

The Mixed Procedure

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Parameter Search

CovP1	CovP2	CovP3	CovP4	CovP5	Variance
0.1000	0	0.002000	0.9000	2.0000	0.7833

Parameter Search

Res Log Like	-2 Res Log Like
-795.8100	1591.6201

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
1	2	1585.88288302	4.32632258
2	1	1477.38038848	0.35523155
3	1	1458.68634705	0.17657682
4	1	1442.12227448	0.06193237
5	1	1435.72757061	0.01884676
6	1	1433.18718904	0.00133908
7	1	1432.97719608	0.00005588
8	1	1432.96646389	0.00000014
9	1	1432.96643725	0.00000000

Convergence criteria met.

Estimated R Matrix for treeid 312

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	4.9817	4.7478	4.5248	4.3124	4.1099	3.9169
2	4.7478	4.9817	4.7478	4.5248	4.3124	4.1099
3	4.5248	4.7478	4.9817	4.7478	4.5248	4.3124
4	4.3124	4.5248	4.7478	4.9817	4.7478	4.5248

ETC

Estimated G Matrix

Row	Effect	treeid	Col1	Col2
1	DBHOB	312	0.06819	-0.00113
2	dbhsq	312	-0.00113	0.000085

Estimated R Matrix for treeid 320

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	4.9817	4.7478	4.5248	4.3124	4.1099	3.9169
2	4.7478	4.9817	4.7478	4.5248	4.3124	4.1099
3	4.5248	4.7478	4.9817	4.7478	4.5248	4.3124
4	4.3124	4.5248	4.7478	4.9817	4.7478	4.5248

ETC…

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	treeid	0.06819
UN(2,1)	treeid	-0.00113
UN(2,2)	treeid	0.000085
SP(POW)	treeid	0.9904
Residual		4.9817

Fit Statistics

-2 Res Log Likelihood	1433.0
AIC (smaller is better)	1443.0
AICC (smaller is better)	1443.1
BIC (smaller is better)	1449.4

PARMS Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
4	158.65	<.0001

The Mixed Procedure

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	2.5901	0.4338	525	5.97	<.0001	0.05
DBHOB	0.9090	0.05921	26	15.35	<.0001	0.05
dbhsq	0.004652	0.002194	26	2.12	0.0437	0.05
dbhcu	-0.00002	0.000013	525	-1.50	0.1342	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	1.7380	3.4422
DBHOB	0.7873	1.0307
dbhsq	0.000142	0.009161
dbhcu	-0.00005	6.259E-6

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.1881	-0.00398	0.000025	-2.24E-7
2	DBHOB	-0.00398	0.003506	-0.00008	2.048E-7
3	dbhsq	0.000025	-0.00008	4.813E-6	-1.21E-8
4	dbhcu	-2.24E-7	2.048E-7	-1.21E-8	1.82E-10

Solution for Random Effects

Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
DBHOB	312	-0.03603	0.08513	525	-0.42	0.6723
dbhsq	312	-0.00500	0.002233	525	-2.24	0.0255
DBHOB	320	0.008269	0.09084	525	0.09	0.9275
dbhsq	320	-0.00650	0.002296	525	-2.83	0.0048
DBHOB	401	-0.1848	0.09434	525	-1.96	0.0507
dbhsq	401	-0.00131	0.002378	525	-0.55	0.5815
DBHOB	407	0.1467	0.1202	525	1.22	0.2227
dbhsq	407	-0.00263	0.003224	525	-0.82	0.4152

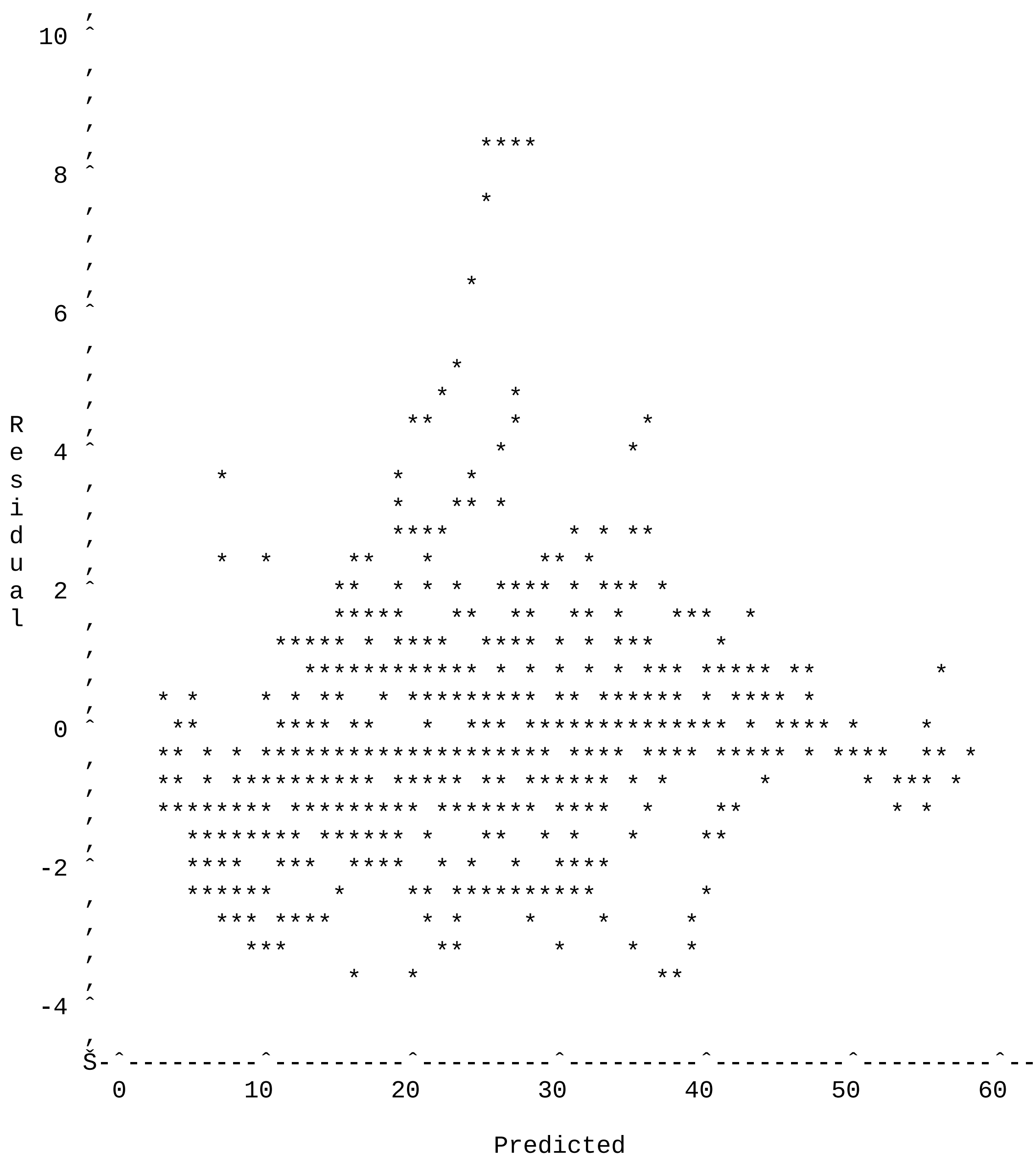
ETC…

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	26	235.69	<.0001
dbhsq	1	26	4.50	0.0437
dbhcu	1	525	2.25	0.1342

MIXED MODEL 4: varying slopes+correlated errors
conditional residuals (only white noise) and predicted values using fixed+random effects

Plot of Resid*Pred. Symbol used is '*'.



NOTE: 206 obs hidden.

MIXED MODEL 4: varying slopes+correlated errors 82
conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Pred*HTTOT. Symbol used is '*'.

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       0        10        20        30        40        50        60

                                   HTTOT
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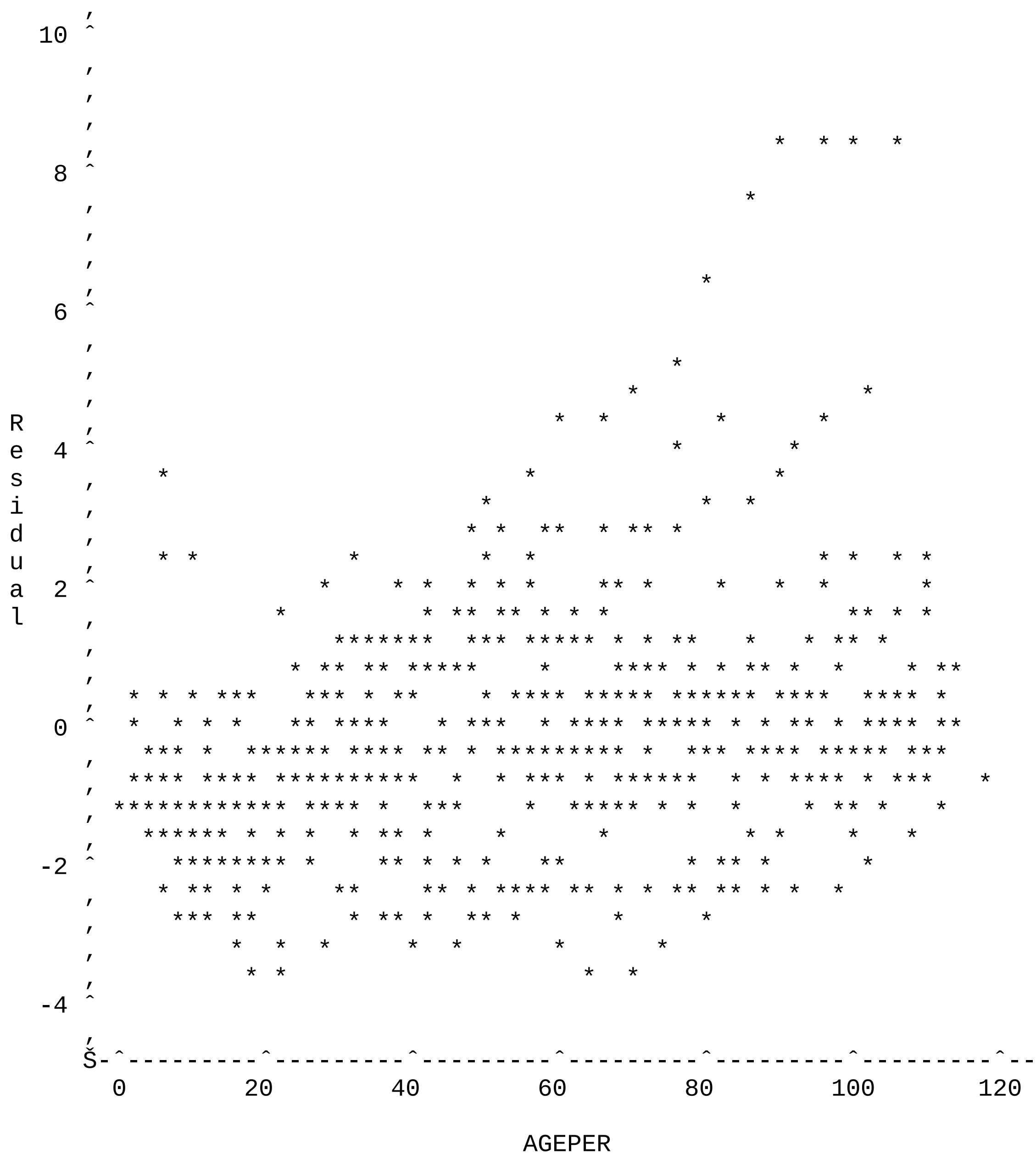
NOTE: 403 obs hidden.

MIXED MODEL 4: varying slopes+correlated errors 83
conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Resid*AGEPER. Symbol used is '*'.



NOTE: 190 obs hidden.
Normality Plot and Marginal residuals not shown.

MIXED MODEL 5: varying slopes+correlated errors 92

The Mixed Procedure

Model Information

Data Set	WORK.SORTED
Dependent Variable	HTTOT
Covariance Structures	Unstructured, Spatial Power
Subject Effects	treeid, treeid
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
treeid	27	312 320 401 407 501 515 518 525 529 603 615 621 626 704 708 709 710 722 802 804 806 810 812 901 906 907 916
AGEPER	116	0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 117

Dimensions

Covariance Parameters	8
Columns in X	4
Columns in Z Per Subject	3
Subjects	27
Max Obs Per Subject	24

The Mixed Procedure

Number of Observations

Number of Observations Read	581
Number of Observations Used	581
Number of Observations Not Used	0

Parameter Search

CovP1	CovP2	CovP3	CovP4	CovP5	CovP6	CovP7
0.2000	0	0.002000	0	0	0	0.9000

Parameter Search

CovP8	Variance	Res Log Like	-2 Res Log Like
2.0000	0.7589	-794.9103	1589.8206

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
1	2	1587.83089596	2.01458762
2	1	1585.59585459	0.00425608
3	1	1582.29128209	0.00633259
4	1	1576.31966208	0.01157594
5	1	1566.45414505	0.01949710
6	1	1556.20925143	0.02066527
7	1	1549.68503587	0.01333568
8	1	1545.57284819	0.00847668
9	1	1536.76650545	0.01848862
10	1	1514.22276627	0.04968123
11	1	1469.33468297	0.10978313

ETC..

Iteration	Evaluations	-2 Res Log Like	Criterion
165	1	1396.58489397	0.00000324
166	1	1396.58381681	0.00000320
167	1	1396.58275090	0.00000317
168	1	1396.58169574	0.00000314
169	1	1396.58065083	0.00000311
170	1	1396.57961571	0.00000308
171	1	1396.57858995	0.00000305
172	1	1396.57757313	0.00000303
173	1	1396.57656484	0.00000300

Convergence criteria met but final hessian is not positive definite.

Estimated R Matrix for treeid 312

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	2.8639	2.6643	2.4785	2.3057	2.1450	1.9954
2	2.6643	2.8639	2.6643	2.4785	2.3057	2.1450
3	2.4785	2.6643	2.8639	2.6643	2.4785	2.3057

ETC…

Estimated G Matrix

Row	Effect	treeid	Col1	Col2	Col3
1	DBHOB	312	0.2853	-0.01997	0.000306
2	dbhsq	312	-0.01997	0.002071	-0.00003
3	dbhcu	312	0.000306	-0.00003	6.022E-7

Estimated R Matrix for treeid 320

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	2.8639	2.6643	2.4785	2.3057	2.1450	1.9954
2	2.6643	2.8639	2.6643	2.4785	2.3057	2.1450
3	2.4785	2.6643	2.8639	2.6643	2.4785	2.3057
4	2.3057	2.4785	2.6643	2.8639	2.6643	2.4785
5	2.1450	2.3057	2.4785	2.6643	2.8639	2.6643
6	1.9954	2.1450	2.3057	2.4785	2.6643	2.8639
7	1.8563	1.9954	2.1450	2.3057	2.4785	2.6643

ETC..

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	treeid	0.2853
UN(2,1)	treeid	-0.01997
UN(2,2)	treeid	0.002071
UN(3,1)	treeid	0.000306
UN(3,2)	treeid	-0.00003
UN(3,3)	treeid	6.022E-7
SP(POW)	treeid	0.9857
Residual		2.8639

Fit Statistics

-2 Res Log Likelihood	1396.6
AIC (smaller is better)	1412.6
AICC (smaller is better)	1412.8
BIC (smaller is better)	1422.9

The Mixed Procedure

PARMS Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
7	193.24	<.0001

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	2.4186	0.3343	499	7.24	<.0001	0.05
DBHOB	0.8176	0.1103	26	7.41	<.0001	0.05
dbhsq	0.01570	0.009176	26	1.71	0.0990	0.05
dbhcu	-0.00027	0.000157	26	-1.75	0.0919	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	1.7619	3.0754
DBHOB	0.5908	1.0443
dbhsq	-0.00316	0.03456
dbhcu	-0.00060	0.000048

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.1117	-0.00396	0.000068	-5.98E-7
2	DBHOB	-0.00396	0.01217	-0.00084	0.000013
3	dbhsq	0.000068	-0.00084	0.000084	-1.42E-6
4	dbhcu	-5.98E-7	0.000013	-1.42E-6	2.466E-8

Solution for Random Effects

Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
DBHOB	312	0.006861	0.1423	499	0.05	0.9616
dbhsq	312	-0.01393	0.009700	499	-1.44	0.1516
dbhcu	312	0.000235	0.000160	499	1.47	0.1412
DBHOB	320	0.1984	0.1381	499	1.44	0.1517
dbhsq	320	-0.02039	0.009444	499	-2.16	0.0313
dbhcu	320	0.000274	0.000158	499	1.74	0.0827

Etc..

MIXED MODEL 5: varying slopes+correlated errors 107

The Mixed Procedure

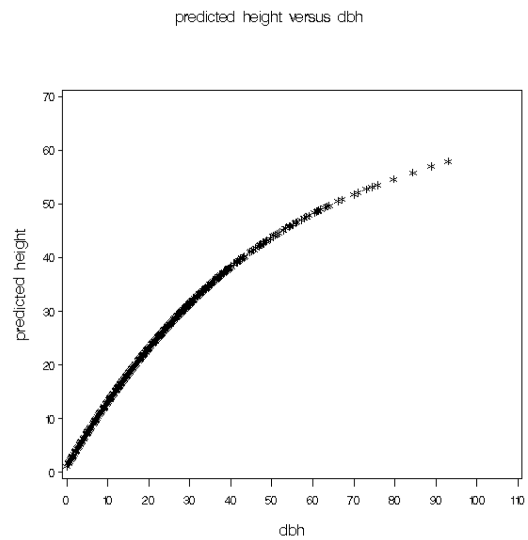
Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
DBHOB	1	26	54.93	<.0001
dbhsq	1	26	2.93	0.0990
dbhcu	1	26	3.06	0.0919

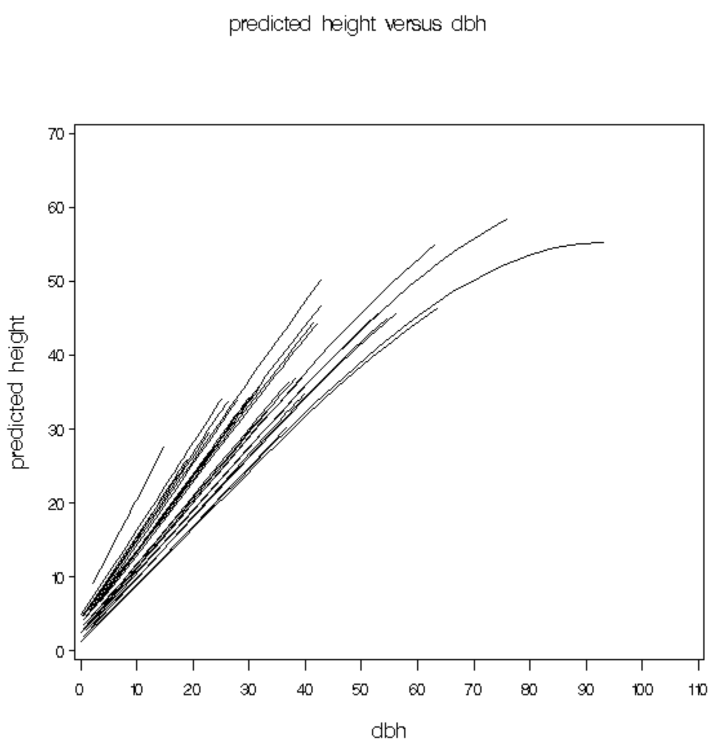
Marginal and Conditional Residuals and other plots not shown..

Fitted Line Graphs:

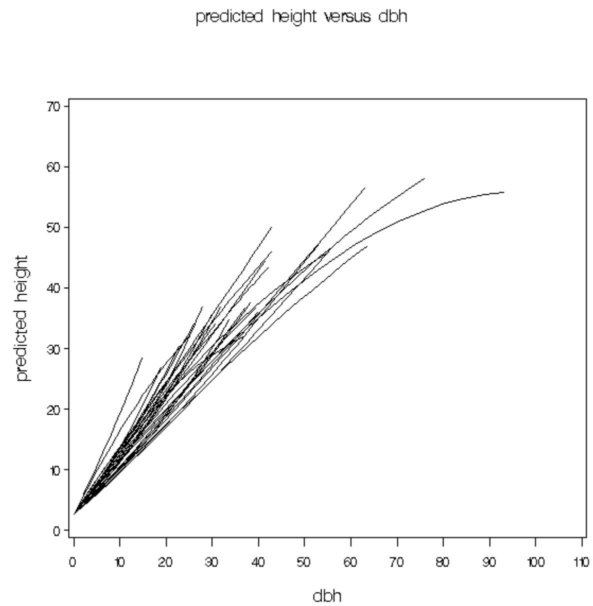
OLS: 1 line for all trees:



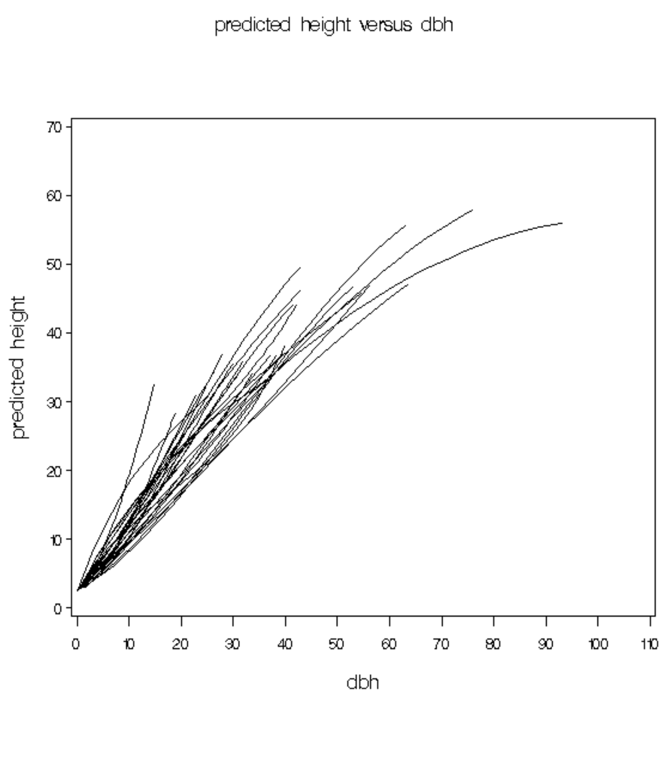
MIXED M3: Correlated Errors, Varying Intercept and Slope with DBHOB:



MIXED M3: Correlated Errors, Varying Slopes with DBHOB and DBHSQ:



MIXED M3: Correlated Errors, Varying Slopes with DBHOB, DBHSQ, DBHCU:

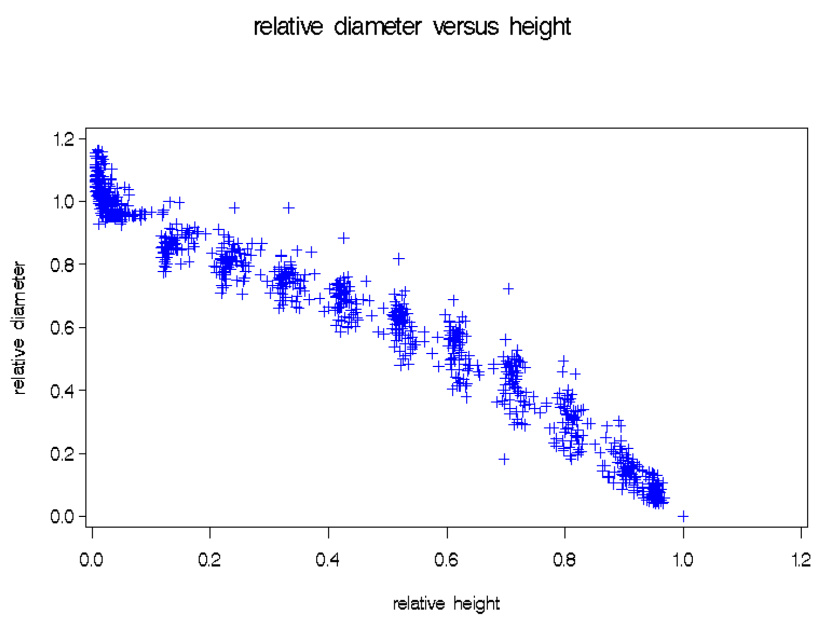


Random Coefficients Modelling

- In a random coefficients model, one or more of the coefficients of the fixed part of the model are considered to have a fixed part and a random part.
- The random part is normally distributed, and may be explained further through other fixed effects.
- When data are in clusters, the fixed component of the model may be varied for each cluster.
- Clusters are class variables (e.g., plots, etc.)
- If all clusters are in the data (e.g., species of interest), simply include this as a class variable in the fixed component of your model
- If only a sample of clusters are in the data (e.g. plots), using a linear mixed model, you can get:
    - a population level model (population average model) with associated confidence intervals for population level coefficients
    - estimates of the distribution (variances) of the coefficients for the cluster, and
    - specific estimates for the clusters in the dataset, along with estimated standard errors.

Example:

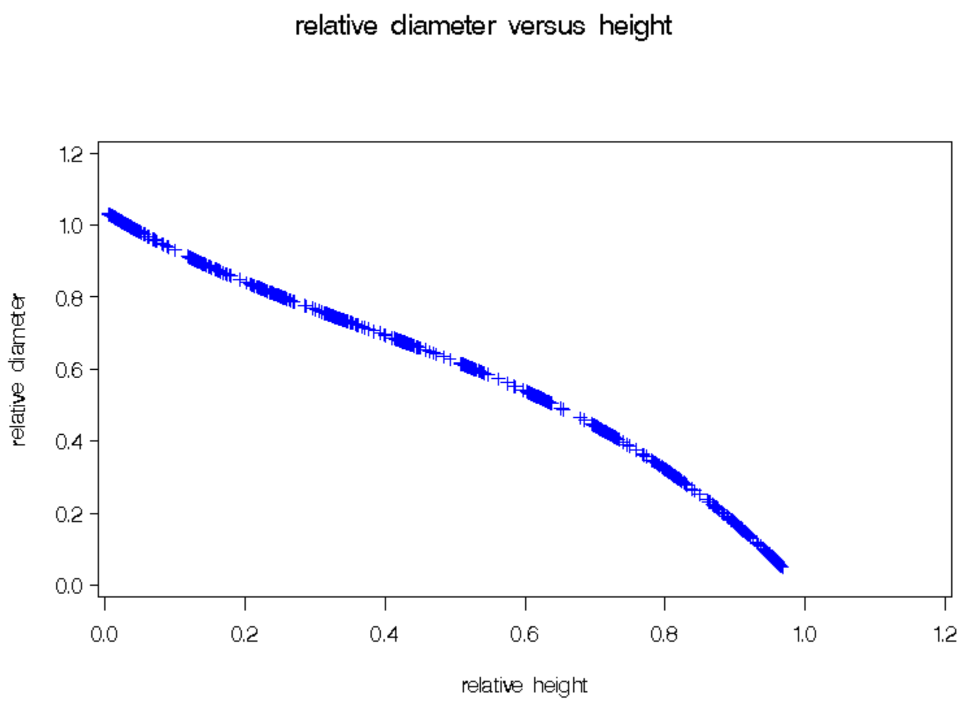
- Have several measures of diameter (dib) at different points (*i*) on the truck (htagr) of several trees (*j*) (taper data).
- These are converted to relative sizes, reldi (di/dbh) and relht (htagr/httot).



- The model to be fitted is:

$$reldi_{ij} = \beta_0 + \beta_1 relht_{ij} + \beta_2 relht^2{}_{ij}$$
$$+ \beta_3 relht^3{}_{ij} + \varepsilon_{ij}$$

Using OLS, the model is fitted to obtain:



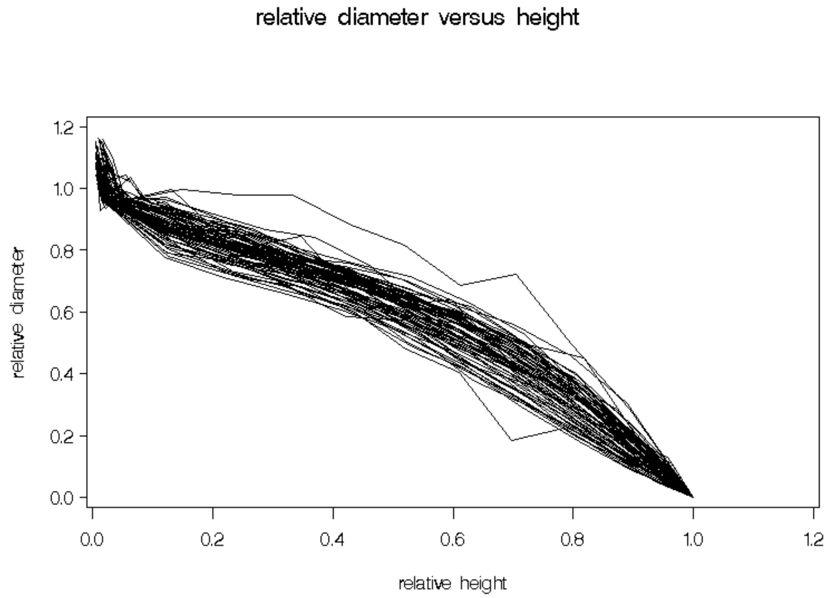
This model was also fitted for each tree as:

$$y_{ij} = \beta_0 + b_{0j} + \beta_1 relht_{ij} + b_{1j} relht_{ij} + \beta_2 relht^2{}_{ij} + b_{2j} relht^2{}_{ij} + \beta_3 relht^3{}_{ij} + b_{3j} relht^3{}_{ij} + \varepsilon_{ij}$$

where b_{0j} is the change in the intercept for subject *j*; b_{1j} , b_{2j} , b_{3j} are the changes in the slopes with for subject *j*. This can be rearranged to separate the fixed effects from the random effects:

$$y_{ij} = \beta_0 + \beta_1 relht_{ij} + \beta_2 relht^2{}_{ij} + \beta_3 relht^3{}_{ij} + b_{0j} + b_{1j} relht_{ij} + b_{2j} relht^2{}_{ij} + b_{3j} relht^3{}_{ij} + \varepsilon_{ij}$$

A graph of the raw data by tree (measures for each tree are connected by lines) is:



Using OLS, a separate function was fitted by tree (same coefficients as including tree as a dummy variable, and including all interactions with the continuous variables). Since there are many trees, the variances and covariances among coefficients were estimated after obtaining all of the tree coefficients.

Variance/covariances of coefficients fitted using OLS by TREEID.

Covariance Matrix, DF = 72

	Intercept	relht	relhtsq	relhtcu
Intercept	0.000918911	-0.005030256	0.008189488	-0.004126866
relht	-0.005030256	0.142995240	-0.347357981	0.211144883
relhtsq	0.008189488	-0.347357981	1.066709904	-0.732375787
relhtcu	-0.004126866	0.211144883	-0.732375787	0.528715529

Using a mixed model, and assuming these are all random coefficients, PROC MIXED was used to obtain estimated coeffcients for each tree, and an estimate of this G matrix of variances /covariances among coefficients, using three alternative methods:

MIXED: method = ML nobound

Estimated G Matrix

Row	Effect	treeid	Col1	Col2	Col3	Col4
1	Intercept	320	0.000465	-0.00155	0.001160	-0.00006
2	relht	320	-0.00155	0.09097	-0.2221	0.1328
3	relhtsq	320	0.001160	-0.2221	0.7406	-0.5187
4	relhtcu	320	-0.00006	0.1328	-0.5187	0.3846

MIXED: method = ML

Estimated G Matrix

Row	Effect	treeid	Col1	Col2	Col3	Col4
1	Intercept	320	0.000465	-0.00155	0.001162	-0.00006
2	relht	320	-0.00155	0.09101	-0.2223	0.1329
3	relhtsq	320	0.001162	-0.2223	0.7412	-0.5191
4	relhtcu	320	-0.00006	0.1329	-0.5191	0.3849

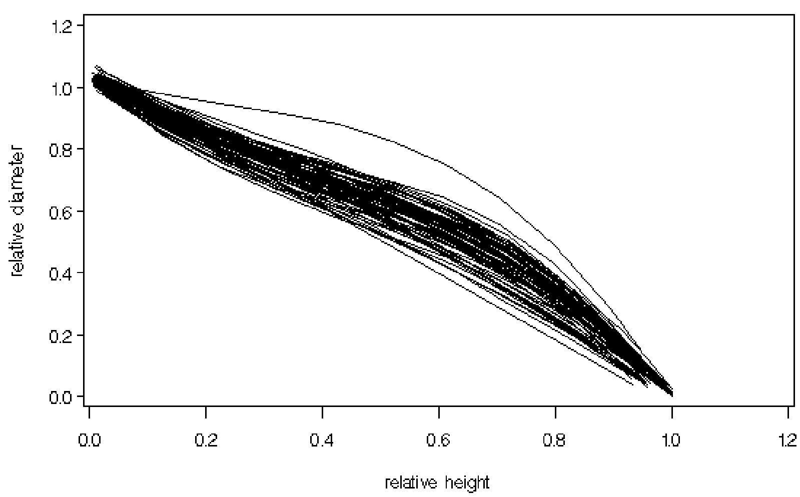
MIXED: method = REML

Estimated G Matrix

Row	Effect	treeid	Col1	Col2	Col3	Col4
1	Intercept	320	0.000477	-0.00162	0.001270	-0.00011
2	relht	320	-0.00162	0.09296	-0.2270	0.1357
3	relhtsq	320	0.001270	-0.2270	0.7557	-0.5291
4	relhtcu	320	-0.00011	0.1357	-0.5291	0.3921

The fit for each tree using PROC MIXED, method=ML, nobound resulted in the following graph of predicted taper for each tree:

pred. relative diameter versus height



Other possibilities:

(1) the data are actually a three-level hierarchical dataset, with measures on trees, and trees in plots. Plot level estimates and variances could also be used.

(2) The within tree spatial correlation could be accounted for as there are many measures per tree, using a spatial correlation:

```
Repeated htagr/subject=tree type=sp(pow)htagr r=1,2;
```

(3) The variation among coefficients by tree could be accounted for adding in tree and plot level explanatory variables (will call these "d"), to create a nonlinear function:

$$y_{ij} = \beta_0 + f_1(d) + \beta_1 relht_{ij} + f_2(d) relht_{ij} + \beta_2 relht^2{}_{ij} + f_3(d) relht^2{}_{ij}$$
$$+ \beta_3 relht^3{}_{ij} + f_4(d) relht^3{}_{ij} + \varepsilon_{ij}$$

Where the variation in coefficients among trees is explained by a function of fixed explanatory variables, plus some left over tree-level variation among coefficients (NOTE: This was called a "parameter prediction approach" in forestry literature prior to Mixed Models).

SAS code: mixed_random_coefficients_taper.sas

```
PROC IMPORT OUT= WORK.taper_data
  DATAFILE= "E:\FRST530\datasets\taper_stand_data.xls"
  DBMS=EXCEL REPLACE;
  SHEET="taper_data";    GETNAMES=YES;    MIXED=NO;
  SCANTEXT=YES;       USEDATE=YES;       SCANTIME=YES;
RUN;
options ls=70 ps=50 pageno=1 nodate;

title1 ' ';

PROC SORT DATA = taper_data out=sorted;
        BY PLOT TREE htagr;
RUN ;

* first graph the scatter of datapoints regardless of tree;
GOPTIONS RESET=ALL;

GOPTIONS DEVICE=WIN; *screen;
* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=2.6 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0 F=SWISS
'relative diameter') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS 'relative
height') ;
```

```
GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL5 C=BLACK V=star;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'relative diameter
versus height' ;
TITLE3 '  ';
*  FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM  ' ' ;
*  FOOTNOTE2 ;
*  FOOTNOTE3 ;

PROC GPLOT DATA = SORTED ;
        PLOT reldi * relht=5/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;

data taper_data2;
set taper_data;
relhtsq=relht**2;
relhtcu=relht**3;
run;

* OLS all trees pooled together;
PROC reg data=taper_data2;
model reldi=relht relhtsq relhtcu;
output out=pout r=resid p=yhat;
run;
proc plot data=pout;
plot yhat*reldi='*';
plot resid*yhat='*';
run;
```

```
* second plot of single fitted line for all trees from OLS
fit;
PROC SORT DATA = pout out=sorted;
  BY PLOT TREE htagr;
RUN ;

GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=2.6 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0
F=SWISS 'relative diameter') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS
'relative height') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL5 C=BLACK V=star;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C
 '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'relative diameter
versus height' ;
TITLE3 '  ';

*  FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM  ' ' ;
*  FOOTNOTE2 ;
*  FOOTNOTE3 ;

PROC GPLOT DATA = SORTED ;
        PLOT yhat * relht=5/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;

PROC SORT DATA = pout out=sorted;
        BY PLOT TREE htagr;
RUN ;
```

```
* Third plot, connect the measures for each tree, by
creating a unique treeid first;
data taper_data3;
set taper_data2;
treeid=(plot*100+tree);
run;
proc sort data=taper_data3 out=sorted3;
by treeid htagr;
run;

GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=2.6 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0
F=SWISS 'relative diameter') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS
'relative height') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=red L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=green L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=blue L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C
 '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'relative diameter
versus height' ;
TITLE3 '  ';

*  FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM  ' ' ;
*  FOOTNOTE2 ;
*  FOOTNOTE3 ;

PROC GPLOT DATA = SORTED3 ;
        PLOT reldi * relht=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;
```

```
* fit a random coefficients model using PROC MIXED, and
including the plot as a random effect, to separate out from
the error term;
proc mixed data=sorted3 method=ml maxiter=100 nobound;
* can alter this to method=ml and the default of bound
also;
title1 'MIXED MODEL: varying slopes';
class treeid;
model reldi = relht relhtsq relhtcu / solution cl covb
residual outp=cond1 residual outpm=marg1 residual;
random intercept relht relhtsq relhtcu/ subject=treeid
type=un solution g;
run;

* use the conditional means out of MIXED first in plotting
(intermediate or narrow)-- only the last error term as the
residual;
proc plot data=cond1;
title2 'conditional residuals (only white noise) and
predicted values';
title3 'using fixed+random effects';
       plot resid*pred='*';
run;
proc univariate data=cond1 plot normal;
          var resid;
run;
* use the marginal means out of MIXED next in plotting
(broad-- population averaged)-- all random components in
residual;
proc plot data=marg1;
title2 'marginal residuals (all random components)';
title3 'and predicted values using fixed effects only';
    plot resid*pred='*';
run;
proc univariate data=marg1 plot normal;
          var resid;
run;
```

```
* plot the fitted lines by tree using the predicted
marginal means;

GOPTIONS RESET=ALL;
GOPTIONS DEVICE=WIN; *screen;

* GOPTIONS DEVICE=WINPRTM;  *hardcopy;

AXIS1 LENGTH=2.6 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM A=90 R=0
F=SWISS 'relative diameter') ;

AXIS2 LENGTH=5 IN MINOR=NONE ORDER=0 TO 1.2 BY 0.2
        VALUE=(H=0.3 CM F=SWISS) LABEL=(H=0.3 CM F=SWISS
'relative height') ;

GOPTIONS RESET=SYMBOL FBY=SWISS HBY=0.35 CM;

SYMBOL1 C=BLACK L=1 V=NONE R=200  I = JOIN ;
SYMBOL2 C=red L=1 V=NONE R=200  I = JOIN ;
SYMBOL3 C=green L=1 V=NONE R=200  I = JOIN ;
SYMBOL4 C=blue L=1 V=NONE R=200  I = JOIN ;

TITLE1 C=BLACK F=SWISS H=0.35 CM J=C
 '  ' ;
TITLE2 C=BLACK F=SWISS H=0.40 CM J=C 'pred. relative
diameter versus height' ;
TITLE3 '  ';

*  FOOTNOTE1 J=C C=BLACK F=SWISS BOX=1 H=0.3 CM  ' ' ;
*  FOOTNOTE2 ;
*  FOOTNOTE3 ;

PROC GPLOT DATA = cond1 ;
        PLOT pred * relht=treeid/
        VAXIS = AXIS1
        HAXIS = AXIS2 NOLEGEND;
RUN ;
```

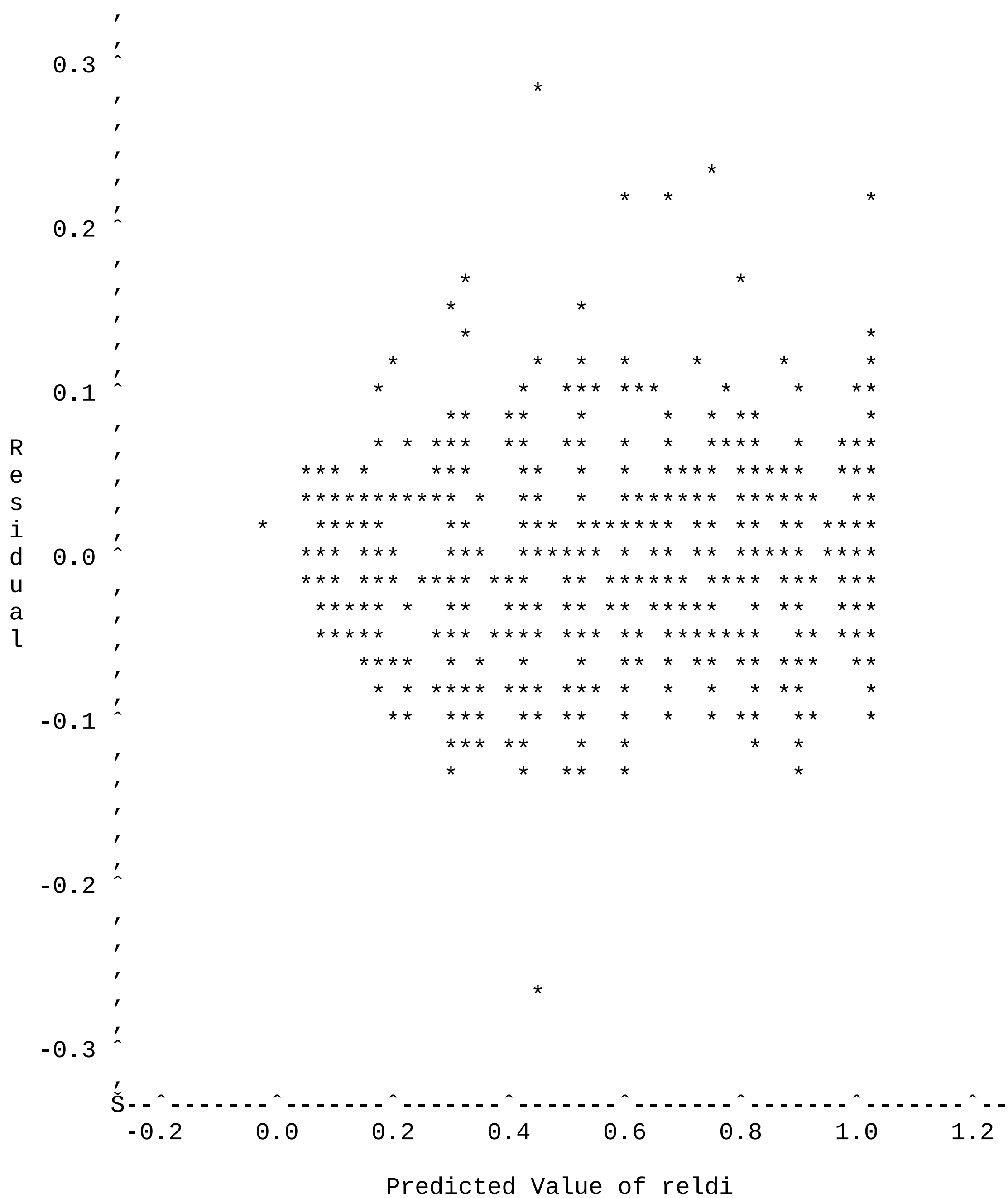
```
/* look at variations in coefficients based on OLS fit of
each tree, but don't print out the results, just pass the
coefficients of each tree to PROC CORR for a summary */

Proc reg data=taper_data3 outest=coeffs noprint;
title1 'OLS by plot -- same as if plot was included as
dummy variables';
title2 'along with all interactions';
model reldi=relht relhtsq relhtcu;
output out=pout2 p=yhat r=resid;
by treeid;
run;

proc corr data=coeffs cov;
var intercept relht relhtsq relhtcu;
with intercept relht relhtsq relhtcu;
run;
```

OUTPUTS:

```
                OLS fit -- all trees combined                     1

                      The REG Procedure
                        Model: MODEL1
               Dependent Variable: reldi reldi

           Number of Observations Read        1084
           Number of Observations Used        1084

                     Analysis of Variance

                                Sum of          Mean
 Source                DF      Squares        Square   F Value  Pr > F
 Model                  3    127.05606      42.35202   15728.7  <.0001
 Error               1080      2.90807       0.00269
 Corrected Total     1083    129.96413


         Root MSE              0.05189    R-Square     0.9776
         Dependent Mean        0.61701    Adj R-Sq     0.9776
         Coeff Var             8.41011


                     Parameter Estimates

                             Parameter     Standard
Variable   Label      DF      Estimate        Error  t Value  Pr > |t|

Intercept  Intercept   1       1.03746      0.00364   284.96    <.0001
relht      relht       1      -1.18660      0.04292   -27.65    <.0001
relhtsq                1       1.28785      0.10933    11.78    <.0001
relhtcu                1      -1.15235      0.07297   -15.79    <.0001
```

```
                    OLS fit -- all trees combined                    2

               Plot of yhat*reldi.  Symbol used is '*'.

      ,
      ,
  1.2 ^
      ,
      ,
      ,
      ,
      ,                                              ************   *
P 1.0 ^                                              **********
r     ,                                             *******
e     ,                                              **
d     ,                                      ************
i     ,                                        * ****
c     ,                                   ***********
t 0.8 ^                                   * *******    *
e     ,                                 ******* ***
d     ,                                 ******** *     *
      ,                              * ***** *  *
V     ,                             *********     *
a     ,                             *  ***  *
l 0.6 ^                        ************    *
u     ,                        * * **
e     ,                    ************* *
      ,                   * *****  ** *
o     ,                  *    **
f     ,          *    ************ *       *
  0.4 ^               * ****   **
r     ,              *  **
e     ,           **********  * *
l     ,          ***********  *
d     ,            ** * *
i     ,        ** * * *
  0.2 ^      *******   *
      ,     ********* *
      ,    *******
      ,   ******
      ,   *****
      ,
  0.0 ^ *
      ,
      Š-^-----------^-----------^-----------^-----------^-----------^-
      0.00        0.25        0.50        0.75        1.00        1.25

                                    reldi

NOTE: 834 obs hidden.
```

Plot of resid*yhat. Symbol used is '*'.



NOTE: 757 obs hidden.

The Mixed Procedure

Model Information

Data Set	WORK.SORTED3
Dependent Variable	reldi
Covariance Structure	Unstructured
Subject Effect	treeid
Estimation Method	ML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
treeid	73	320 621 624 626 704 708 722 812 916 1007 1118 1401 1618 1805 1810 1910 2010 2206 2311 2705 2706 2806 2811 2814 2817 3001 3101 3109 3301 3311 3407 3503 3605 3608 3804 3906 3917 4001 4005 4209 4305 4407 4411 4601 4608 4706 4801 4807 4809 4811 4908 4909 4912 5009 5102 5103 5204 5211 5305 5311 5402 5404 5510 5606 5607 5909 5914 6002 6007 6201 6311 6315 6401

Dimensions

Covariance Parameters	11
Columns in X	4
Columns in Z Per Subject	4
Subjects	73
Max Obs Per Subject	15

Number of Observations

Number of Observations Read	1084
Number of Observations Used	1084
Number of Observations Not Used	0

The Mixed Procedure

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	-3342.02217467	
1	2	-4055.32770351	0.00000552
2	1	-4055.34473559	0.00000001

Convergence criteria met.

Estimated G Matrix

Row	Effect	treeid	Col1	Col2	Col3	Col4
1	Intercept	320	0.000465	-0.00155	0.001160	-0.00006
2	relht	320	-0.00155	0.09097	-0.2221	0.1328
3	relhtsq	320	0.001160	-0.2221	0.7406	-0.5187
4	relhtcu	320	-0.00006	0.1328	-0.5187	0.3846

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	treeid	0.000465
UN(2,1)	treeid	-0.00155
UN(2,2)	treeid	0.09097
UN(3,1)	treeid	0.001160
UN(3,2)	treeid	-0.2221
UN(3,3)	treeid	0.7406
UN(4,1)	treeid	-0.00006
UN(4,2)	treeid	0.1328
UN(4,3)	treeid	-0.5187
UN(4,4)	treeid	0.3846
Residual		0.001034

Fit Statistics

-2 Log Likelihood	-4055.3
AIC (smaller is better)	-4025.3
AICC (smaller is better)	-4024.9
BIC (smaller is better)	-3991.0

The Mixed Procedure

Null Model Likelihood Ratio Test

DF	Chi-Square	Pr > ChiSq
10	713.32	<.0001

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Alpha
Intercept	1.0378	0.003404	72	304.90	<.0001	0.05
relht	-1.2083	0.04434	72	-27.25	<.0001	0.05
relhtsq	1.3528	0.1216	72	11.12	<.0001	0.05
relhtcu	-1.1970	0.08564	72	-13.98	<.0001	0.05

Solution for Fixed Effects

Effect	Lower	Upper
Intercept	1.0310	1.0446
relht	-1.2967	-1.1199
relhtsq	1.1103	1.5953
relhtcu	-1.3677	-1.0263

Covariance Matrix for Fixed Effects

Row	Effect	Col1	Col2	Col3	Col4
1	Intercept	0.000012	-0.00007	0.000106	-0.00005
2	relht	-0.00007	0.001966	-0.00481	0.002936
3	relhtsq	0.000106	-0.00481	0.01480	-0.01017
4	relhtcu	-0.00005	0.002936	-0.01017	0.007334

Solution for Random Effects

Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	320	-0.00617	0.01332	792	-0.46	0.6434
relht	320	-0.5066	0.1507	792	-3.36	0.0008
relhtsq	320	0.8150	0.3354	792	2.43	0.0153
relhtcu	320	-0.2981	0.1587	792	-1.88	0.0607

The Mixed Procedure

Solution for Random Effects

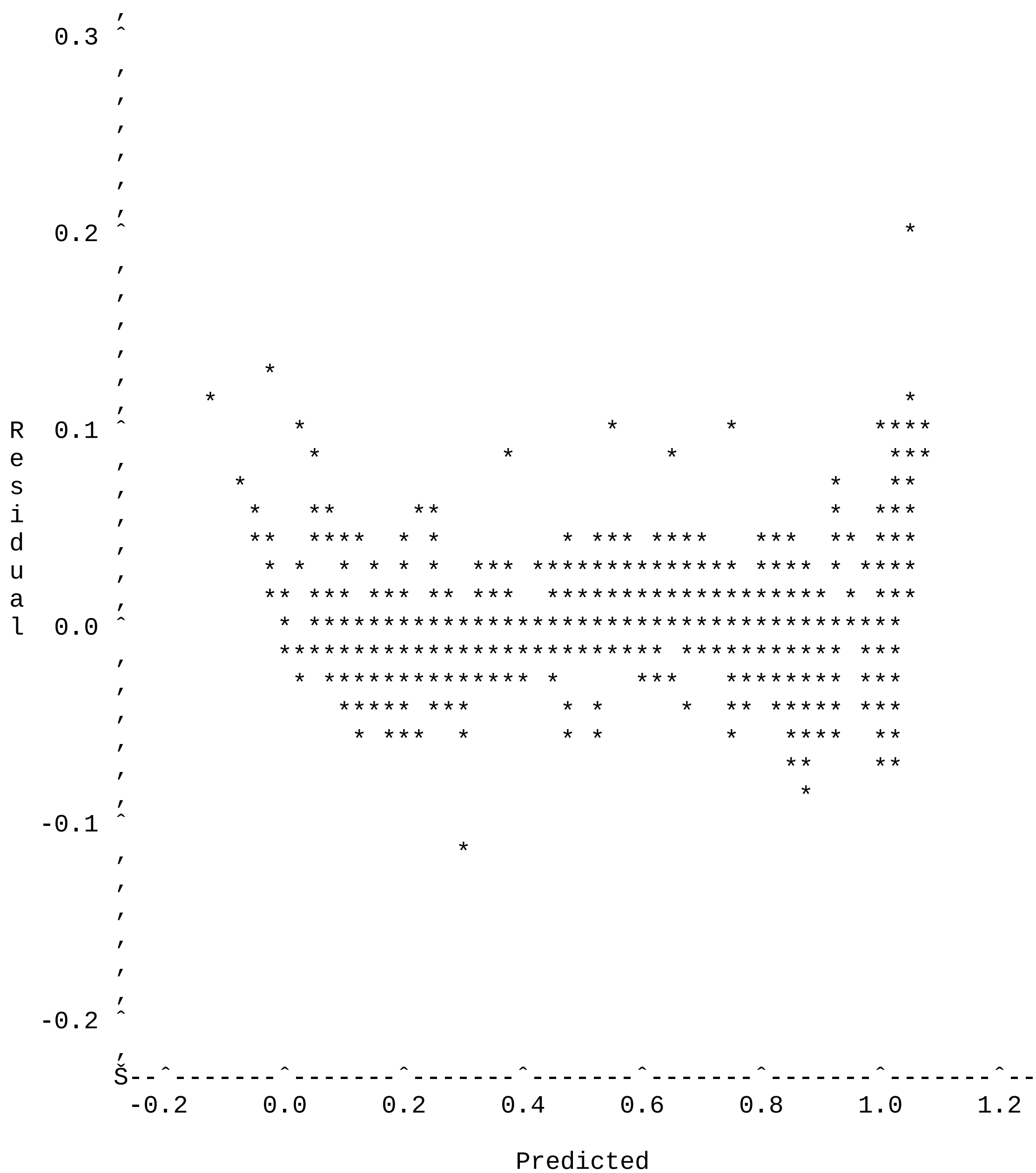
Effect	treeid	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	621	-0.00173	0.01369	792	-0.13	0.8996
relht	621	-0.2012	0.1542	792	-1.30	0.1924
relhtsq	621	0.6886	0.3473	792	1.98	0.0478
relhtcu	621	-0.4593	0.1730	792	-2.66	0.0081
Intercept	624	-0.01987	0.01406	792	-1.41	0.1579
relht	624	0.03035	0.1562	792	0.19	0.8460
relhtsq	624	0.4159	0.3541	792	1.17	0.2405
relhtcu	624	-0.4042	0.1831	792	-2.21	0.0276
(some outputs deleted – all values given for each tree)						
Intercept	6311	0.006499	0.01506	792	0.43	0.6663
relht	6311	0.2865	0.1606	792	1.78	0.0748
relhtsq	6311	-1.2625	0.3627	792	-3.48	0.0005
relhtcu	6311	0.9790	0.1932	792	5.07	<.0001
Intercept	6315	-0.00772	0.01483	792	-0.52	0.6029
relht	6315	-0.02441	0.1587	792	-0.15	0.8778
relhtsq	6315	-0.6853	0.3532	792	-1.94	0.0527
relhtcu	6315	0.7293	0.1801	792	4.05	<.0001
Intercept	6401	0.01481	0.01526	792	0.97	0.3320
relht	6401	0.3206	0.1584	792	2.02	0.0433
relhtsq	6401	-1.3793	0.3434	792	-4.02	<.0001
relhtcu	6401	1.0334	0.1646	792	6.28	<.0001

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
relht	1	72	742.50	<.0001
relhtsq	1	72	123.67	<.0001
relhtcu	1	72	195.36	<.0001

conditional residuals (only white noise) and predicted values
using fixed+random effects

Plot of Resid*Pred. Symbol used is '*'.



NOTE: 810 obs hidden.

```
                     MIXED MODEL: varying slopes                    16
    conditional residuals (only white noise) and predicted values
                      using fixed+random effects

                       The UNIVARIATE Procedure
                     Variable:  Resid  (Residual)
```

(some outputs deleted)

```
                         Tests for Normality

      Test                  --Statistic---    -----p Value------
      Shapiro-Wilk          W     0.962642    Pr < W     <0.0001
      Kolmogorov-Smirnov    D     0.067893    Pr > D     <0.0100
      Cramer-von Mises      W-Sq  0.996939    Pr > W-Sq  <0.0050
      Anderson-Darling      A-Sq  6.766982    Pr > A-Sq  <0.0050

                            Normal Probability Plot
         0.21+                                                  *
             |
             |
             |
             |                                                  *
             |                                                 **
             |                                             *****
             |                                          ****+++++
         0.05+                                      +****++
             |                                +*******
             |                         +********
             |                  *********
             |           ********
             |    ********+
             |****++
             |*
        -0.11+*
              +----+----+----+----+----+----+----+----+----+----+
                  -2        -1         0        +1        +2
```

MIXED MODEL: varying slopes
marginal residuals (all random components)
and predicted values using fixed effects only

Plot of Resid*Pred. Symbol used is '*'.

```
   0.3 ^
       ,                            *
       ,
       ,
       ,                                        *
       ,                                  *  *             *
   0.2 ^
       ,
       ,                       *                  *
       ,                      *        *
       ,                       *               *           *
       ,                  *         *  *  *    *     *     *
   0.1 ^                 *         *   ** ***    *    *   **
       ,                      **   *   **    *  ** *       *
R      ,                 * * ***  **  **  ** ** ****  *  ***
e      ,            *** *    ***   **  *  *  * ********  ***
s      ,            ******** ** *  **  *  ******* ******  **
i      ,         *   *****   ***   *** ******* ** ** ** ****
d  0.0 ^            *** ***   ** * ****** * ** ** ***** ****
u      ,            *** ***  *** **** ** ****** **** **  ***
a      ,             *****  * **   ** ** ** ***** ** **  ***
l      ,             *****   *** **** *** ** *******  *  ***
       ,                ****  * *  *   *  ** * **  * **   **
       ,                   * **** * * *** *  *  *  * **   *
  -0.1 ^                 ***  ***  ** **  *  *  * **  *    *
       ,                      *** **   *  *        *  *
       ,                      *    *  **  *           *
       ,
       ,
       ,
  -0.2 ^
       ,
       ,
       ,
       ,                            *
       ,
  -0.3 ^
       Š--^-------^-------^-------^-------^-------^-------^-------^--
        -0.2     0.0     0.2     0.4     0.6     0.8     1.0     1.2

                               Predicted Mean
```

NOTE: 761 obs hidden.

```
                 MIXED MODEL: varying slopes                  20
          marginal residuals (all random components)
        and predicted values using fixed effects only

                   The UNIVARIATE Procedure
                 Variable:  Resid  (Residual)
```

(some outputs deleted)

```
                      Tests for Normality

Test                  --Statistic---    -----p Value------
Shapiro-Wilk          W      0.97644    Pr < W     <0.0001
Kolmogorov-Smirnov    D     0.051101    Pr > D     <0.0100
Cramer-von Mises      W-Sq  0.637142    Pr > W-Sq  <0.0050
Anderson-Darling      A-Sq  3.905114    Pr > A-Sq  <0.0050


                       Normal Probability Plot
  0.275+                                                 *
       |                                                 *
       |                                                **
       |                                           ******+
       |                                  +********+
       |                       *************
       |              **********
       |    ***********
       |******
       |
       |
 -0.275+*
        +----+----+----+----+----+----+----+----+----+----+
            -2        -1         0        +1        +2
```

```
OLS by plot -- same as if plot was included as dummy variables  23
                 along with all interactions
```

(NOTE: results not printed –coefficients passed to PROC CORR for analysis)

```
                      The CORR Procedure

  4 With Variables:    Intercept relht     relhtsq   relhtcu
  4      Variables:    Intercept relht     relhtsq   relhtcu

                  Covariance Matrix, DF = 72

                                     Intercept             relht
 Intercept      Intercept          0.000918911      -0.005030256
 relht          relht             -0.005030256       0.142995240
 relhtsq                           0.008189488      -0.347357981
 relhtcu                          -0.004126866       0.211144883
```

Covariance Matrix, DF = 72

		relhtsq	relhtcu
Intercept	Intercept	0.008189488	-0.004126866
relht	relht	-0.347357981	0.211144883
relhtsq		1.066709904	-0.732375787
relhtcu		-0.732375787	0.528715529

Simple Statistics

Variable	N	Mean	Std Dev	Sum
Intercept	73	1.03712	0.03031	75.71007
relht	73	-1.20860	0.37815	-88.22756
relhtsq	73	1.35920	1.03282	99.22182
relhtcu	73	-1.20263	0.72713	-87.79179

Simple Statistics

Variable	Minimum	Maximum	Label
Intercept	0.96681	1.12503	Intercept
relht	-2.06536	-0.26950	relht
relhtsq	-0.67208	3.59461	
relhtcu	-2.66552	0.28578	

OLS by plot -- same as if plot was included as dummy variables 24
along with all interactions

The CORR Procedure

Pearson Correlation Coefficients, N = 73
Prob > |r| under H0: Rho=0

	Intercept	relht	relhtsq	relhtcu
Intercept	1.00000	-0.43883	0.26158	-0.18723
Intercept		0.0001	0.0254	0.1127
relht	-0.43883	1.00000	-0.88939	0.76791
relht	0.0001		<.0001	<.0001
relhtsq	0.26158	-0.88939	1.00000	-0.97521
	0.0254	<.0001		<.0001
relhtcu	-0.18723	0.76791	-0.97521	1.00000
	0.1127	<.0001	<.0001	

Generalized Linear Models

Generalized linear models are an extension of linear models that allows for:

- A nonlinear <u>link</u> function, which is a model that transforms the *y*, in order to predict the transformed y with a linear function of the *x*'s.
- Response probability distributions that can be any member of the exponential family of <u>distributions</u> (e.g., normal, inverse normal, gamma, binomial, negative binomial, Poisson and multinomial).
- Unequal variances of the *y*'s (variance of the *y*'s is a function of the mean of *y*, given *x*)
- Errors are uncorrelated for ease of calculation of the likelihood

NOTE: If errors are correlated, Generalized Estimating Equations (GEEs) can be used

Many widely used statistical models are generalized linear models, including:

- classic linear models with normal errors,
- logistic and probit models for binary data
- Poisson regression
- Log-linear models for multinomial data. Many other useful

Many other statistical models can be formulated as generalized linear models by the selection of an appropriate <u>link</u> function and response probability <u>distribution</u>.

For the OLS model:

$$y_i = \mathbf{x_i}'\boldsymbol{\beta} + \varepsilon_i$$

$$\varepsilon_i \text{ are iid and} \sim N(0, {\sigma_\varepsilon}^2)$$

$$\mu_i = \mu_y \mid x_i = \mathbf{x_i}'\boldsymbol{\beta}$$

NOTE: In this notation, the *x* values for a particular observation are given as a column vector of values.

For generalized linear models, this is extended to:

$$\eta = \mathbf{x_i}'\boldsymbol{\beta} = g(\mu_i)$$

$$y_i = g^{-1}(\mu_i) + \varepsilon_i$$

$$\text{var}(y_i) = \frac{\phi V(\mu_i)}{w_i}$$

$$y_i \sim \text{any of the family of exponential distribution}$$

$$y_i \text{ are independent (or can use GEE)}$$

- ϕ is a constant, called the scale parameter
- V is a variance function (variance as some function of the mean)
- w_i is a weight for each observation (default is each observation is given a weight=1.

Distributions, SAS Default Link Functions, and Variance Functions, Using PROC GENMOD

Distribution DIST=	Default SAS Link Function LINK=	Variance Functions
Normal	Identity $g(\mu_i) = \mu_i$ (no transformation to the y)	$V(\mu_i) = 1$ (equal variance)
Binomial (proportion)	Logit $g(\mu_i) = \ln\left(\dfrac{\mu_i}{1-\mu_i}\right)$	$V(\mu_i) = \mu_i(1-\mu_i)$ (like p(1-p) for the binomial)
Poisson	Inverse (Power(-1)) $g(\mu_i) = \mu_i^{-1}$	$V(\mu_i) = \mu_i$ (variance is equal to the mean for Poisson)
Gamma		$V(\mu_i) = \mu_i^{2}$
Inverse Gaussian	Inverse Squared (Power(-2)) $g(\mu_i) = \mu_i^{-2}$	$V(\mu_i) = \mu_i^{3}$
Negative Binomial	Log $g(\mu_i) = \ln(\mu_i)$	$V(\mu_i) = \mu_i + k\mu_i^{2}$

- See Schabenberger and Pierce and also Wikipedia for good descriptions of distributions in the exponential family and what processes they might represent.
- Other link functions that can be used include: Other powers, the logarithm and probit as an alternative to logit (see SAS documentation)

Commonly Used Generalized Linear Models

1. OLS model: y is continuous, LINK=identity, DIST=normal

2. Logistic Regression: y is a proportion (or a 0,1 Bernoulli variable), LINK=logit, DIST=binomial [can use PROC LOGISTIC instead]

3. Poisson Regression, log linear model: y is a count (no natural denominator, else use y as a proportion), LINK=log, DIST=Poisson

4. Count using Negative Binomial: y is a count (no natural denominator, else use y as a proportion), LINK=log, DIST=negbin

5. Gamma Model with log linear model: y is a positive continuous variable, DIST=gamma, LINK=log.

Under/Overdispersion
If the default variance for the specified distribution does not match the data then the data are *over-* or *underdispersed.* This can happen with the binomial and Poisson distributions. An overdispersion factor can be added to the variance function and an estimate of this found by MLE along with the other parameters. Alternatively, another distribution may be more appropriate (e.g., switch to negative binomial for count data).

Model Goodness of Fit

1. Grouped data: Use deviances

   =2 (ln likelihood for best possible model – ln likelihood for model fitted) X scale parameter

   Where the best possible model is where $y_i = y_i$.

   OR Pearson's Chi Squared Statistic [see section 6.4.3 of Schabenberger & Pierce, and SAS documentation on how these are calculated for the distribution specified for the model]
2. Better models have higher likelihood (or log likelihood), which is the same as saying the -2 lnL is smaller.
3. Pseudo R squared value, based on lnL of the model versus lnL for a "null model" with only the intercept (no explanatory variables). Several forms of these have been developed, in order to obtain a similar interpretation to R^2 for linear models.

Comparing Nested Models

- Use Deviance or Pearson's Chi Squared Statistic, for grouped or ungrouped data to compare unrestricted to restricted models, called, "Deviance partitioning", using a likelihood ratio test.
- This is like extra sums of squares used in partial F-tests for least squared fitted models.

[Schabenberger & Pierce, section 6.4.4]

Generalized Estimating Equations (GEE)
When data are correlated, GEE can be used. Essentially, this is iterated EGLS. The steps are:

1. Use OLS to obtain the estimated coefficients for the linear model of transformed y (via the link) versus x's.
2. Use the standardized residuals from OLS to estimate the error covariance matrix, based on a specific type of correlation (e.g., AR(1) is one type, unstructured is another).
3. Use EGLS to calculate the estimated coefficients.
4. Use standardized residuals from EGLS to estimate the error covariance matrix.
5. Repeat steps 3 and 4 until convergence occurs (iterated EGLS – will be the maximum likelihood solution).

Generalized Linear Mixed Models

Generalized linear mixed models are an extension of general linear models.

Recall that generalized linear models allow for:

- A nonlinear link function, which is a model that transforms the *y*, in order to predict the transformed y with a linear function of the *x*'s
- Response probability distributions that can be any member of the exponential family of distributions (e.g., normal, inverse normal, gamma, binomial, negative binomial, Poisson and multinomial).
- Unequal variances of the *y*'s (variance of the *y*'s is a function of the mean of *y*, given *x*)
- Errors are uncorrelated for ease of calculation of the likelihood

NOTE: If errors are correlated, Generalized Estimating Equations (GEEs) can be used

Extending this to generalized linear mixed models allows for:

- more random-effects to be added (the **Zu** of mixed linear models)
- more complex error structures (the **R** of mixed linear models), including hierarchical structures, where the error covariance matrix is a block diagonal matrix.

The generalized linear model is:

$$\eta = \mathbf{x}_i\boldsymbol{\beta} = g(\mu_i)$$

$$y_i = g^{-1}(\mu_i) + \varepsilon_i \qquad\qquad \mathrm{var}(y_i) = \frac{\phi V(\mu_i)}{w_i}$$

$$y_i \sim \text{any of the family of exponential distribution}$$

$$y_i \text{ are independent (or can use GEE)}$$

- ϕ is a constant, called the scale parameter
- V is a variance function (variance as some function of the mean)
- w_i is a weight for each observation (default is each observation is given a weight=1.

The maximum likelihood fit of this model can be obtained by:

$$\mathbf{X'WX\beta} = \mathbf{X'Wy}^*$$

$$\boldsymbol{\beta} = (\mathbf{X'WX})^{-1}\mathbf{X'Wy}^*$$

$$\text{Same as}: \boldsymbol{\beta} = (\mathbf{X'}\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X'}\Sigma^{-1}\mathbf{y}^*$$

$$\text{the equation for GLS, but using } \mathbf{y}^* \text{ instead of } \mathbf{y}$$

These y* values are pseudo-data. The pseudo-data can be created as a Taylor series expansion of the mean around the current estimate of $\boldsymbol{\beta}$.

The Taylor series is a linear model of derivatives:

$$y^* = f(\hat{\eta}) + (y-\hat{\eta})f'(\hat{\eta}) + \frac{(y-\hat{\eta})^2}{2!}f''(\hat{\eta}) + \frac{(y-\hat{\eta})^3}{3!}f'''(\hat{\eta}) + \ldots$$

$$+ \frac{(y-\hat{\eta})^n}{n!}f^{(n)}(\hat{\eta}) + \ldots$$

The y* depends on the estimate of $\boldsymbol{\beta}$, and so does, W, and, in turn, they depend on y*. Therefore, an iterative search is needed. However, this becomes a linear model, with unequal variances, which can be fitted using least squares methods, and then iterated to a maximum likelihood solution, when the estimate of $\boldsymbol{\beta}$ no longer changes (Pseudo-Likelihood).

The generalized linear mixed model is:

$$\eta = \mathbf{x}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{u} = g(\mu_i)$$

$$y_i = g^{-1}(\mu_i) + \varepsilon_i$$

Random effects (the $\mathbf{u}$) ARE assumed to be normally distributed with mean 0

The Taylor series expansion could then be around the conditional means OR around the marginal means, and a Pseudo-Likelihood search performed.

Solution in SAS
PROC GLIMMIX: SAS uses this approach of Pseudo-Likelihood (PL) and either the conditional means (with random effects) which they term the "subject level" or the marginal (fixed effects only) means (S or M). There is also Maximum likelihood Versus Restricted Maximum Likelihood version (M or R). The methods are then: RSPL or MSPL for subject-level pseudo-likelihood, or RMPL or MMPL for population level (conditional) pseudo-likelihood.

NOTE: GLIMMIX is a special "add-on" to SAS at present. You must download the software and add it to your SAS.

Example: We have measured number of regenerated trees per ha in each plot, along with plot competition measures, baha and ccf. We also have the class variable site_srs, indicating moisture and nutrient levels, where a 1 is a model (medium) site. There are nonlinear relationships, and transformations of baha and ccf are included. (NOTE: 25% of the data are selected for use)

Run 1: Use GENMOD, link=log dist=binomial dscale

Run 2: Use GLIMMIX method=mmpl, link=log dist=binomial and let the intercept vary with site_srs

Run 3: Use GLIMMIX method=mspl, link=log dist=binomial and let the intercept vary with site_srs

Run 4: Use GLIMMIX method=mspl, link=log dist=binomial and let the intercept and slope with baha vary with site_srs

Run 5: Use GLIMMIX method=mspl, link=log dist=binomial and include site_srs as a random-effects class variable (exactly the same as Run 3)

Can get residuals and predicted values by adding in:

```
output out=glimmix pred(blup ilink)=pcond resid(blup
ilink)=rcond
pred(noblup ilink)=pmarg resid(noblup ilink)=rmarg;
```

SAS code:

```
* import plot data for IDF;
PROC IMPORT OUT= WORK.PLtdata
DATAFILE= "E:\fii_regen_data\IDF_data\Plot-for-grph-
correl.xls"
DBMS=EXCEL REPLACE;     SHEET="Sheet1$";    GETNAMES=YES;
MIXED=NO;   SCANTEXT=YES;   USEDATE=YES;  SCANTIME=YES;
RUN;
* select 25 percent of the data for use;
Data PLtdata2;
set Pltdata;
random=rand('Uniform');
if random > 0.25 then delete;
bahasq=baha_all**2;
ccfsq=ccf**2;
keep tph_all baha_all bahasq ccf ccfsq site_srs;
run;

options ps=50 ls=64 pageno=1 nodate;
proc plot data=PLtdata2;
Title1 'basic plots of data';
plot tph_all*(baha_all bahasq ccf ccfsq)=site_srs;
run;
* MODEL 1: use deviance to estimate the scale parameter;
proc genmod data=PLtdata2;
Title1 'log link Poisson';
  model tph_all=baha_all bahasq ccf ccfsq/link=log
dist=poisson dscale obstats;
  ODS output obstats=check;
run;
data graphdat;
merge check PLtdata2;
run;

proc plot data=graphdat;
Title1 'log link Poisson';
 plot resdev*xbeta=site_srs;
 plot resraw*pred=site_srs;
 plot tph_all*xbeta=site_srs;
 plot pred*tph_all=site_srs;
 run;
 * glimmix with site_srs brought in as a random intercept;
proc glimmix data=PLtdata2 method=mmpl;
Title1 'log link Poisson random intercept by site_srs-mmpl';
class site_srs;
 model tph_all=baha_all bahasq ccf ccfsq/link=log
dist=poisson;
random int/type=UN subject=site_srs solution cl;
output out=glimmix1 pred(blup ilink)=pcond resid(blup
ilink)=rcond
 pred(noblup ilink)=pmarg resid(noblup ilink)=rmarg;
run;
```

```
 * glimmix with site_srs brought in as a random intercept;
proc glimmix data=PLtdata2 method=mspl;
Title1 'log link Poisson random intercept by site_srs-mspl';
class site_srs;
 model tph_all=baha_all bahasq ccf ccfsq/link=log
dist=poisson;
random int/type=UN subject=site_srs solution cl;
output out=glimmix2 pred(blup ilink)=pcond resid(blup
ilink)=rcond
 pred(noblup ilink)=pmarg resid(noblup ilink)=rmarg;
run;

 * glimmix with site_srs brought in as a random intercept;
proc glimmix data=PLtdata2 method=mspl;
Title1 'log link Poisson random intercept slope baha_all by
site series-mspl';
class site_srs;
 model tph_all=baha_all bahasq ccf ccfsq/link=log
dist=poisson;
random int baha_all/type=UN subject=site_srs solution cl;
output out=glimmix3 pred(blup ilink)=pcond resid(blup
ilink)=rcond
 pred(noblup ilink)=pmarg resid(noblup ilink)=rmarg;
run;
 * glimmix with site_srs brought in as a random class
variable - same as random intercept;
proc glimmix data=PLtdata2 method=mspl;
Title1 'log link Poisson random site_srs -- same as random
int by site series-mspl';
class site_srs;
 model tph_all=baha_all bahasq ccf ccfsq/link=log
dist=poisson;
random site_srs/solution cl;
output out=glimmix4 pred(blup ilink)=pcond resid(blup
ilink)=rcond
 pred(noblup ilink)=pmarg resid(noblup ilink)=rmarg;
run;
```

```
                         basic plots of data                          1

       Plot of TPH_ALL*BAHA_ALL.  Symbol is value of Site_srs.

        ,
        ,
  10000 ^
        ,
        ,
        ,
        ,                              5
        ,
        ,
   8000 ^
        ,
        ,
        ,
        ,
        ,                                                        1
        ,      1
T  6000 ^                                     1
P       ,
H       ,                     1
_       ,
A       ,
L       ,
L       ,
   4000 ^                     1            1
        ,                                  1
        ,                                          1
        ,   1         1                                     1
        ,                      1    1    5
        ,                      11
        ,                      4     1  3
   2000 ^  1  5          5           1    1  1
        ,                44           3 1 1         4
        ,       3 5     1              5
        ,   4    1        141 115 4 45      5
        , 1      1   1 1134    1  14   5  5  1
        ,1     4    715  4   85
        ,5    1    8         5       5      4
      0 ^4       5      1
        ,
        Š^---------^---------^---------^---------^---------^---------^
         0        10        20        30        40        50        60

                                   BAHA_ALL

NOTE: 8 obs hidden.
```

Plot of TPH_ALL*CCF. Symbol is value of Site_srs.

```
            ,
            ,
      10000 ˆ
            ,
            ,
            ,
            ,                          5
            ,
            ,
       8000 ˆ
            ,
            ,
            ,
            ,
            ,                                              1
            ,       1
    T  6000 ˆ                             1
    P       ,
    H       ,                  1
    _       ,
    A       ,
    L       ,
    L       ,
       4000 ˆ               1         1
            ,                             1
            ,                       1
            ,     1        1                     1
            ,                  1 51
            ,                1  1
            ,                1 41     3
       2000 ˆ   1  5       5 1     1     1
            ,            4  4 3   1 1      4
            ,        5  1         5
            ,     4 1     14 14 5   5
            ,    1 1  113  4111 4  5
            ,  1  47 5 1815
            ,  5  1 8      55  4
          0 ˆ  4   5 1
            ,
            Š--ˆ-----------ˆ-----------ˆ-----------ˆ-----------ˆ--
               0          100         200         300         400

                                      CCF
```

NOTE: 14 obs hidden.

(remainder of basic plots, not shown)

The GENMOD Procedure

Model Information

Data Set	WORK.PLTDATA2	
Distribution	Poisson	
Link Function	Log	
Dependent Variable	TPH_ALL	TPH_ALL

Number of Observations Read	86
Number of Observations Used	86

Criteria For Assessing Goodness Of Fit

Criterion	DF	Value	Value/DF
Deviance	81	41840.5660	516.5502
Scaled Deviance	81	81.0000	1.0000
Pearson Chi-Square	81	49237.1844	607.8665
Scaled Pearson X2	81	95.3193	1.1768
Log Likelihood		1850.4516	

Algorithm converged.

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square
Intercept	1	6.8656	0.1959	6.4817	7.2494	1228.79
BAHA_ALL	1	-0.2280	0.0476	-0.3213	-0.1347	22.92
bahasq	1	0.0021	0.0008	0.0006	0.0036	7.17
CCF	1	0.0464	0.0075	0.0317	0.0611	38.39

Analysis Of Parameter Estimates

Parameter	Pr > ChiSq
Intercept	<.0001
BAHA_ALL	<.0001
bahasq	0.0074
CCF	<.0001

The GENMOD Procedure

Analysis Of Parameter Estimates

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square
ccfsq	1	-0.0001	0.0000	-0.0001	-0.0000	12.40
Scale	0	22.7277	0.0000	22.7277	22.7277	

Analysis Of Parameter Estimates

Parameter	Pr > ChiSq
ccfsq	0.0004
Scale	

NOTE: The scale parameter was estimated by the square root of DEVIANCE/DOF.

Observation Statistics

Observation	TPH_ALL	BAHA_ALL Pred Lower Resdev	bahasq Xbeta Upper StResdev	CCF Std Resraw StReschi	ccfsq HessWgt Reschi Reslik
1	375	35.3	1246.09	135	18225
		628.66551	6.4435993	0.200828	1.2170463
		424.10615	931.89012	-253.6655	-10.117
		-10.94651	-0.493911	-0.456483	-0.49214
2	1101	27.9	778.41	127.6	16281.76
		1045.7523	6.9524918	0.1153159	2.024493
		834.20296	1310.9494	55.247716	1.7084407
		1.6937195	0.075546	0.0762026	0.0755637
3	851	7.6	57.76	37.4	1398.76
		987.49341	6.8951698	0.1168689	1.9117085
		785.33549	1241.69	-136.4934	-4.343547
		-4.449856	-0.198397	-0.193657	-0.198275

(remainder of OBSTATS not shown; not all plots shown)

```
          Plot of Resdev*Xbeta.  Symbol is value of Site_srs.
Resdev ,
   100 ˆ
       ,
       ,                               1
       ,
    80 ˆ
       ,
       ,
       ,
    60 ˆ
       ,
       ,                           1    1
       ,
    40 ˆ                    1                               5
       ,                              1
       ,                             1   5            1
       ,                       3    5            1    1
    20 ˆ                           1   1
       ,                            4   1
       ,                    5    1                           1
       ,               7 1 5     1 5     1
     0 ˆ               8   5  1  4   3  1
       ,                5 4    41 1  1   5    54    1
       ,                 4       53  15 11         11
       ,        1        55         1    44   1
   -20 ˆ                            1 1   55   4     1   1  1
       ,                        15        44       4
       ,                  5     5     1              13
       ,
   -40 ˆ
       ,                        4
       ,
       ,
   -60 ˆ
       ,
       Š--ˆ-------ˆ-------ˆ-------ˆ-------ˆ-------ˆ-------ˆ-------ˆ--
         5.5      6      6.5      7      7.5      8      8.5      9
                                    Xbeta
NOTE: 9 obs hidden.
```

```
         Plot of Pred*TPH_ALL.  Symbol is value of Site_srs.
Pred ,
6000 ˆ
     ,                                          1
     ,                                                       5
     ,
     ,                        1
     ,
5000 ˆ
     ,
     ,
     ,                    1
     ,
     ,
4000 ˆ               3
     ,                                  1  1
     ,             1
     ,                1 1
     ,                   1
     ,            4     1
3000 ˆ
     ,                          1
     ,
     ,           4  4
     ,           1  5
     ,
2000 ˆ        5
     ,      444
     ,       415   11    5     1
     ,    1   1  1   1                      1
     ,      1453  4    1 1
     ,     1   5  1 5      1
1000 ˆ  451 11141  3
     ,     41      1
     ,  545515
     ,    87
     ,  1
     ,
   0 ˆ
     ,
     Š-ˆ-----------ˆ-----------ˆ-----------ˆ-----------ˆ-----------ˆ--
       0          2000        4000        6000        8000        10000

                                  TPH_ALL

NOTE: 19 obs hidden.
```

The GLIMMIX Procedure

Model Information

Data Set	WORK.PLTDATA2
Response Variable	TPH_ALL
Response Distribution	Poisson
Link Function	Log
Variance Function	Default
Variance Matrix Blocked By	Site_srs
Estimation Technique	MPL
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
Site_srs	6	1 3 4 5 7 8

Number of Observations Read	86
Number of Observations Used	86

Dimensions

G-side Cov. Parameters	1
Columns in X	5
Columns in Z per Subject	1
Subjects (Blocks in V)	6
Max Obs per Subject	45

Optimization Information

Optimization Technique	Dual Quasi-Newton
Parameters in Optimization	1
Lower Boundaries	1
Upper Boundaries	0
Fixed Effects	Profiled
Starting From	Data

The GLIMMIX Procedure

Iteration History

Iteration	Restarts	Subiterations	Objective Function	Change
0	0	4	43963.603856	0.46505817
1	0	3	49222.160619	0.18163592
2	0	2	49692.995165	0.01349634
3	0	2	49688.253217	0.00045681
4	0	1	49687.447123	0.00010143
5	0	1	49687.809342	0.00002154
6	0	1	49687.715922	0.00001360
7	0	1	49687.737152	0.00000297
8	0	1	49687.73266	0.00000878
9	0	1	49687.7339	0.00002630
10	0	2	49687.732712	0.00001975
11	0	0	49687.733501	0.00000000

Convergence criterion (PCONV=1.11022E-8) satisfied.

Fit Statistics

-2 Log Pseudo-Likelihood	49687.73
Generalized Chi-Square	50110.87
Gener. Chi-Square / DF	582.68

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error
UN(1,1)	Site_srs	0.03810	0.02239

Type III Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
BAHA_ALL	1	76	9458.28	<.0001
bahasq	1	76	3019.77	<.0001
CCF	1	76	15811.9	<.0001
ccfsq	1	76	5302.14	<.0001

Solution for Random Effects

Effect	Subject	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	Site_srs 1	0.2072	0.08033	76	2.58	0.0118
Intercept	Site_srs 3	-0.04526	0.08089	76	-0.56	0.5774
Intercept	Site_srs 4	-0.2401	0.08045	76	-2.99	0.0038
Intercept	Site_srs 5	0.1144	0.08041	76	1.42	0.1587
Intercept	Site_srs 7	0.2156	0.08788	76	2.45	0.0165
Intercept	Site_srs 8	-0.2518	0.08391	76	-3.00	0.0036

Solution for Random Effects

Effect	Subject	Alpha	Lower	Upper
Intercept	Site_srs 1	0.05	0.04718	0.3671
Intercept	Site_srs 3	0.05	-0.2064	0.1158
Intercept	Site_srs 4	0.05	-0.4004	-0.07991
Intercept	Site_srs 5	0.05	-0.04570	0.2746
Intercept	Site_srs 7	0.05	0.04056	0.3906
Intercept	Site_srs 8	0.05	-0.4189	-0.08468

log link Poisson random intercept by site_srs-mspl 28

The GLIMMIX Procedure

Model Information

Data Set	WORK.PLTDATA2
Response Variable	TPH_ALL
Response Distribution	Poisson
Link Function	Log
Variance Function	Default
Variance Matrix Blocked By	Site_srs
Estimation Technique	PL
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
Site_srs	6	1 3 4 5 7 8

Number of Observations Read	86
Number of Observations Used	86

Dimensions

G-side Cov. Parameters	1
Columns in X	5
Columns in Z per Subject	1
Subjects (Blocks in V)	6
Max Obs per Subject	45

Optimization Information

Optimization Technique	Dual Quasi-Newton
Parameters in Optimization	1
Lower Boundaries	1
Upper Boundaries	0
Fixed Effects	Profiled
Starting From	Data

The GLIMMIX Procedure

Iteration History

Iteration	Restarts	Subiterations	Objective Function	Change
0	0	4	34699.034818	0.18031469
1	0	2	41689.259291	0.04877062
2	0	2	42846.090387	0.00155506
3	0	1	42875.135096	0.00000225
4	0	1	42875.155731	0.00000541
5	0	1	42875.155726	0.00000000

Convergence criterion (PCONV=1.11022E-8) satisfied.

Fit Statistics

-2 Log Pseudo-Likelihood	42875.16
Generalized Chi-Square	43303.77
Gener. Chi-Square / DF	503.53

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error
UN(1,1)	Site_srs	0.04133	0.02426

Type III Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
BAHA_ALL	1	76	10604.9	<.0001
bahasq	1	76	3411.84	<.0001
CCF	1	76	17836.8	<.0001
ccfsq	1	76	6060.13	<.0001

Solution for Random Effects

Effect	Subject	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	Site_srs 1	0.2080	0.08359	76	2.49	0.0150
Intercept	Site_srs 3	-0.02664	0.08417	76	-0.32	0.7525
Intercept	Site_srs 4	-0.2549	0.08378	76	-3.04	0.0032
Intercept	Site_srs 5	0.1281	0.08367	76	1.53	0.1299
Intercept	Site_srs 7	0.2157	0.08969	76	2.41	0.0186
Intercept	Site_srs 8	-0.2704	0.08815	76	-3.07	0.0030

Solution for Random Effects

Effect	Subject	Alpha	Lower	Upper
Intercept	Site_srs 1	0.05	0.04155	0.3745
Intercept	Site_srs 3	0.05	-0.1943	0.1410
Intercept	Site_srs 4	0.05	-0.4217	-0.08800
Intercept	Site_srs 5	0.05	-0.03853	0.2948
Intercept	Site_srs 7	0.05	0.03709	0.3943
Intercept	Site_srs 8	0.05	-0.4459	-0.09481

log link Poisson random intercept slope baha_all by site series-ms 33

The GLIMMIX Procedure

Model Information

Data Set	WORK.PLTDATA2
Response Variable	TPH_ALL
Response Distribution	Poisson
Link Function	Log
Variance Function	Default
Variance Matrix Blocked By	Site_srs
Estimation Technique	PL
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
Site_srs	6	1 3 4 5 7 8

Number of Observations Read	86
Number of Observations Used	86

Dimensions

G-side Cov. Parameters	3
Columns in X	5
Columns in Z per Subject	2
Subjects (Blocks in V)	6
Max Obs per Subject	45

Optimization Information

Optimization Technique	Dual Quasi-Newton
Parameters in Optimization	3
Lower Boundaries	2
Upper Boundaries	0
Fixed Effects	Profiled
Starting From	Data

log link Poisson random intercept slope baha_all by site series-ms 34

The GLIMMIX Procedure

Iteration History

Iteration	Restarts	Subiterations	Objective Function	Change
0	0	7	33758.746896	0.18831200
1	0	4	40919.076202	0.05477655
2	0	3	42186.82006	0.00199234
3	0	1	42223.872348	0.00000201
4	0	1	42223.908966	0.00000100
5	0	2	42223.90899	0.00000051
6	0	1	42223.908969	0.00000000

Convergence criterion (PCONV=1.11022E-8) satisfied.

Fit Statistics

-2 Log Pseudo-Likelihood	42223.91
Generalized Chi-Square	42620.60
Gener. Chi-Square / DF	495.59

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error
UN(1,1)	Site_srs	0.3326	0.2020
UN(2,1)	Site_srs	-0.01565	0.009926
UN(2,2)	Site_srs	0.000794	0.000506

Type III Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
BAHA_ALL	1	4	311.83	<.0001
bahasq	1	72	3416.85	<.0001
CCF	1	72	17443.4	<.0001
ccfsq	1	72	5724.20	<.0001

Solution for Random Effects

Effect	Subject	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	Site_srs 1	0.4242	0.2385	72	1.78	0.0796
BAHA_ALL	Site_srs 1	-0.01194	0.01190	72	-1.00	0.3188
Intercept	Site_srs 3	0.2905	0.2400	72	1.21	0.2301
BAHA_ALL	Site_srs 3	-0.01604	0.01194	72	-1.34	0.1834
Intercept	Site_srs 4	-0.1159	0.2389	72	-0.49	0.6290
BAHA_ALL	Site_srs 4	-0.00847	0.01191	72	-0.71	0.4793
Intercept	Site_srs 5	0.06626	0.2389	72	0.28	0.7823
BAHA_ALL	Site_srs 5	-0.00037	0.01191	72	-0.03	0.9750
Intercept	Site_srs 7	0.5181	0.2822	72	1.84	0.0705
BAHA_ALL	Site_srs 7	-0.02247	0.01784	72	-1.26	0.2119
Intercept	Site_srs 8	-1.1831	0.2567	72	-4.61	<.0001
BAHA_ALL	Site_srs 8	0.05930	0.01308	72	4.53	<.0001

Solution for Random Effects

Effect	Subject	Alpha	Lower	Upper
Intercept	Site_srs 1	0.05	-0.05137	0.8997
BAHA_ALL	Site_srs 1	0.05	-0.03566	0.01177
Intercept	Site_srs 3	0.05	-0.1879	0.7690
BAHA_ALL	Site_srs 3	0.05	-0.03985	0.007765
Intercept	Site_srs 4	0.05	-0.5920	0.3603
BAHA_ALL	Site_srs 4	0.05	-0.03221	0.01527
Intercept	Site_srs 5	0.05	-0.4099	0.5424
BAHA_ALL	Site_srs 5	0.05	-0.02411	0.02336
Intercept	Site_srs 7	0.05	-0.04448	1.0806
BAHA_ALL	Site_srs 7	0.05	-0.05804	0.01310
Intercept	Site_srs 8	0.05	-1.6949	-0.6713
BAHA_ALL	Site_srs 8	0.05	0.03323	0.08537

log link Poisson random site_srs -- same as random int by site seri 38

The GLIMMIX Procedure

Model Information

Data Set	WORK.PLTDATA2
Response Variable	TPH_ALL
Response Distribution	Poisson
Link Function	Log
Variance Function	Default
Variance Matrix	Not blocked
Estimation Technique	PL
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
Site_srs	6	1 3 4 5 7 8

Number of Observations Read	86
Number of Observations Used	86

Dimensions

G-side Cov. Parameters	1
Columns in X	5
Columns in Z	6
Subjects (Blocks in V)	1
Max Obs per Subject	86

Optimization Information

Optimization Technique	Dual Quasi-Newton
Parameters in Optimization	1
Lower Boundaries	1
Upper Boundaries	0
Fixed Effects	Profiled
Starting From	Data

Iteration History

Iteration	Restarts	Subiterations	Objective Function	Change
0	0	4	34699.034818	0.18031469
1	0	2	41689.259291	0.04877062
2	0	1	42846.090387	0.00155506
3	0	1	42875.135098	0.00000250
4	0	1	42875.155728	0.00000599
5	0	1	42875.155733	0.00001436
6	0	1	42875.15572	0.00001012
7	0	0	42875.155729	0.00000000

Convergence criterion (PCONV=1.11022E-8) satisfied.

Fit Statistics

-2 Log Pseudo-Likelihood	42875.16
Generalized Chi-Square	43303.77
Gener. Chi-Square / DF	503.53

log link Poisson random site_srs -- same as random int by site seri 42

The GLIMMIX Procedure

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error
Site_srs	0.04133	0.02426

Type III Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
BAHA_ALL	1	76	10604.9	<.0001
bahasq	1	76	3411.84	<.0001
CCF	1	76	17836.8	<.0001
ccfsq	1	76	6060.13	<.0001

Solution for Random Effects

Effect	Site_srs	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Site_srs	1	0.2080	0.08359	76	2.49	0.0150
Site_srs	3	-0.02664	0.08417	76	-0.32	0.7525
Site_srs	4	-0.2549	0.08378	76	-3.04	0.0032
Site_srs	5	0.1281	0.08367	76	1.53	0.1299
Site_srs	7	0.2157	0.08969	76	2.41	0.0186
Site_srs	8	-0.2704	0.08815	76	-3.07	0.0030

Solution for Random Effects

Effect	Site_srs	Alpha	Lower	Upper
Site_srs	1	0.05	0.04155	0.3745
Site_srs	3	0.05	-0.1943	0.1410
Site_srs	4	0.05	-0.4217	-0.08800
Site_srs	5	0.05	-0.03853	0.2948
Site_srs	7	0.05	0.03709	0.3943
Site_srs	8	0.05	-0.4459	-0.09481

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